

# The classification of CMC foliations of $\mathbb{R}^3$ and $\mathbb{S}^3$ with countably many singularities

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## Abstract

In this paper we generalize the Local Removable Singularity Theorem in [16] for minimal laminations to the case of weak  $H$ -laminations (with  $H \in \mathbb{R}$  constant) in a punctured ball of a Riemannian three-manifold. We also obtain a curvature estimate for any weak CMC foliation (with possibly varying constant mean curvature from leaf to leaf) of a compact Riemannian three-manifold  $N$  with boundary solely in terms of a bound of the absolute sectional curvature of  $N$  and of the distance to the boundary of  $N$ . We then apply these results to classify weak CMC foliations of  $\mathbb{R}^3$  and  $\mathbb{S}^3$  with a closed countable set of singularities.

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## 1 Introduction.

In this paper, we address a number of outstanding classical questions on the geometry of embedded surfaces of constant mean curvature and more generally, laminations and foliations of  $\mathbb{R}^3$  and other three-manifolds, where the leaves of these laminations are surfaces with constant mean curvature (possibly varying from leaf to leaf). In the foliation case, we call every such foliation a *CMC foliation*. The first of these classical problems is to classify codimension one CMC foliations of  $\mathbb{R}^3$  or  $\mathbb{S}^3$  (with their standard metrics) in the complement of a closed countable set  $\mathcal{S}$ . The simplest such examples in  $\mathbb{R}^3$  are families of parallel planes or concentric spheres

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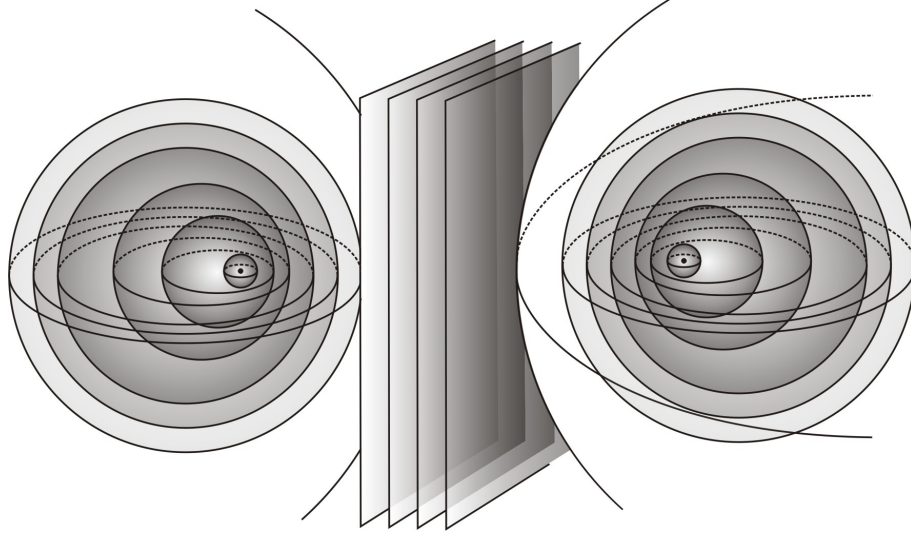


Figure 1: A foliation of  $\mathbb{R}^3$  by spheres and planes with two singularities.

around a given point. A slightly more complicated example appears when considering a family of pairwise disjoint planes and spheres as in Figure 1, where the set  $\mathcal{S}$  consists of two points. We solve this classification problem in complete generality (see Theorem 6.1 for a solution of an even more general problem, where the leaves of the "foliation" are allowed to intersect in a controlled manner<sup>1</sup>):

**Theorem 1.1** *Suppose  $\mathcal{F}$  is a CMC foliation of  $\mathbb{R}^3$  with a closed countable set  $\mathcal{S}$  of singularities<sup>2</sup>. Then, all leaves of  $\mathcal{F}$  are contained in planes and round spheres. Furthermore if  $\mathcal{S}$  is empty, then  $\mathcal{F}$  is a foliation by planes.*

In the case of the unit three-sphere  $\mathbb{S}^3 \subset \mathbb{R}^4$  with its standard metric of constant sectional curvature, we obtain a similar result:

*The leaves of every CMC foliation of  $\mathbb{S}^3$  with a closed countable set  $\mathcal{S}$  of singularities are contained in round spheres, and  $\mathcal{S}$  is always non-empty.*

We remark that the special case of Theorem 1.1 where the singular set  $\mathcal{S}$  of  $\mathcal{F}$  is empty is a classical result of Meeks [12]. Also, we note that in the statement of the above theorem, we made no assumption on the regularity of the foliation  $\mathcal{F}$ . However, the proofs in this paper require that

<sup>1</sup>The quotes here refer to the notion of *weak CMC foliation*; see Definition 3.2.

<sup>2</sup>We mean that  $\mathcal{F}$  is a foliation of  $\mathbb{R}^3 - \mathcal{S}$ , and it does not extend to a CMC foliation of  $\mathbb{R}^3 - \mathcal{S}'$  for any proper closed subset  $\mathcal{S}' \subset \mathcal{S}$ .

$\mathcal{F}$  has bounded second fundamental form on compact sets of  $N = \mathbb{R}^3$  or  $\mathbb{S}^3$  minus the singular set  $\mathcal{S}$ . This bounded curvature assumption always holds for a topological CMC foliation<sup>3</sup> by recent work of Meeks and Tinaglia [19] on curvature estimates for embedded, non-zero constant mean curvature disks and a related 1-sided curvature estimate for embedded surfaces of any constant mean curvature; in the case that all of the leaves of the lamination of a three-manifold are minimal, this 1-sided curvature estimate was given earlier by Colding and Minicozzi [5].

Consider a foliation  $\mathcal{F}$  of a Riemannian three-manifold  $N$  with leaves having constant absolute mean curvature. After possibly passing to a four-sheeted cover, we can assume  $N$  is oriented and that all leaves of  $\mathcal{F}$  are oriented consistently, in the sense that there exists a continuous, nowhere zero vector field in  $N$  which is transversal to the leaves of  $\mathcal{F}$ . In this situation, the mean curvature function of the leaves of  $\mathcal{F}$  is well-defined and so  $\mathcal{F}$  is a CMC foliation. Therefore, when analyzing the structure of such a CMC foliation  $\mathcal{F}$ , it is natural to consider for each  $H \in \mathbb{R}$ , the subset  $\mathcal{F}(H)$  of  $\mathcal{F}$  of those leaves that have mean curvature  $H$ . Such a subset  $\mathcal{F}(H)$  is closed since the mean curvature function is continuous on  $\mathcal{F}$ ;  $\mathcal{F}(H)$  is an example of an  $H$ -lamination. A cornerstone in proving the above classification results is to analyze the structure of an  $H$ -lamination  $\mathcal{L}$  (or more generally, a weak  $H$ -lamination, see Definition 3.2) of a punctured ball in a Riemannian three-manifold, in a small neighborhood of the puncture. This local problem can be viewed as a desingularization problem. In our previous paper [16], we characterized the removability of an isolated singularity of a minimal lamination  $\mathcal{L}$  of a punctured ball in terms of the growth of the norm of the second fundamental form of the leaves of  $\mathcal{L}$  when extrinsically approaching the puncture. We will extend this Local Removable Singularity Theorem to the case of a weak  $H$ -lamination, see Theorem 1.2 below.

Next we set some specific notation to be used throughout the paper, which is necessary to state the next Local Removable Singularity Theorem. Given a Riemannian three-manifold  $N$  and a point  $p \in N$ , we denote by  $d_N$  the distance function in  $N$  and by  $B_N(p, r)$ ,  $\bar{B}_N(p, r)$ ,  $S_N^2(p, r)$  the open metric ball of center  $p$  and radius  $r > 0$ , its closure and boundary sphere, respectively. In the case  $N = \mathbb{R}^3$ , we use the notation  $\mathbb{B}(p, r) = B_{\mathbb{R}^3}(p, r)$ ,  $\mathbb{S}^2(p, r) = S_{\mathbb{R}^3}^2(p, r)$  and  $\mathbb{B}(r) = \mathbb{B}(\vec{0}, r)$ ,  $\mathbb{S}^2(r) = \mathbb{S}^2(\vec{0}, r)$ , where  $\vec{0} = (0, 0, 0)$ . Furthermore,  $R: \mathbb{R}^3 \rightarrow \mathbb{R}$  stands for the distance function to the origin  $\vec{0}$ . For a codimension-one lamination  $\mathcal{L}$  of  $N$  and a leaf  $L$  of  $\mathcal{L}$ , we denote by  $|\sigma_L|$  the norm of the second fundamental form of  $L$ . Since leaves of  $\mathcal{L}$  do not intersect, it makes sense to consider the norm of the second fundamental form as a function defined on the union of the leaves of  $\mathcal{L}$ , which we denote by  $|\sigma_{\mathcal{L}}|$ . In the case of a weak  $H$ -lamination  $\mathcal{L}$  of  $N$ , given  $p \in \mathcal{L}$  there exist at most two leaves of  $\mathcal{L}$  passing through  $p$  (by the maximum principle for constant mean curvature surfaces), and thus, we define  $|\sigma_{\mathcal{L}}|$  to be the function on  $\mathcal{L}$  that assigns to each  $p \in \mathcal{L}$  the maximum of the norms of the second fundamental forms of leaves  $L$  of  $\mathcal{L}$  such that  $p \in L$ . Observe that  $|\sigma_{\mathcal{L}}|$  is not necessarily continuous.

**Theorem 1.2 (Local Removable Singularity Theorem)** *A weak  $H$ -lamination  $\mathcal{L}$  of a punctured ball  $B_N(p, r) - \{p\}$  of a Riemannian three-manifold  $N$  extends to a weak  $H$ -lamination*

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<sup>3</sup>See Definition 3.1 for this concept.

of  $B_N(p, r)$  if and only if there exists a positive constant  $C$  such that  $|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C$  in some subball. In particular under this hypothesis,

1. The second fundamental form of  $\mathcal{L}$  is bounded in a neighborhood of  $p$ .
2. If  $\mathcal{L}$  consists of a single leaf  $M \subset B_N(p, r) - \{p\}$  which is a properly immersed weak  $H$ -surface<sup>4</sup> with  $\emptyset \neq \partial M \subset \partial B_N(p, r)$ , then  $M$  extends smoothly across  $p$ .

We remark that in the case  $H = 0$ , a weak  $H$ -lamination is just a minimal lamination, see the first paragraph just after Definition 3.2. In this way, Theorem 1.2 generalizes the minimal case of the Local Removable Singularity Theorem proven in [16].

Besides the above Local Removable Singularity Theorem, a second key ingredient is needed in the proof of Theorem 1.1: a universal scale-invariant curvature estimate valid for any CMC foliation of a compact Riemannian three-manifold with boundary, solely in terms of an upper bound for its sectional curvature. The next result is inspired by previous curvature estimates by Schoen [24] and Ros [22] for compact stable minimal surfaces with boundary, and by Rosenberg, Souam and Toubiana [23] for stable constant mean curvature surfaces.

**Theorem 1.3 (Curvature Estimates for CMC foliations)** *There exists a positive constant  $A > 0$  such that the following statement holds. Given  $\Lambda \geq 0$ , a compact Riemannian three-manifold  $N$  with boundary whose absolute sectional curvature is at most  $\Lambda$ , a weak CMC foliation  $\mathcal{F}$  of  $N$  and a point  $p \in \text{Int}(N)$ , we have*

$$|\sigma_{\mathcal{F}}|(p) \leq \frac{A}{\min\{\text{dist}_N(p, \partial N), \frac{\pi}{\sqrt{\Lambda}}\}},$$

where  $|\sigma_{\mathcal{F}}|: N \rightarrow [0, \infty)$  is the function that assigns to each  $p \in N$  the supremum of the norms of the second fundamental forms of leaves of  $\mathcal{F}$  passing through  $p$ .

The above curvature estimate is an essential tool for analyzing the structure of a weak CMC foliation of a small geodesic Riemannian three-ball punctured at its center. Among other things we prove that if the mean curvatures of the leaves of such a weak CMC foliation are bounded in a neighborhood of the puncture, then the weak CMC foliation extends across the puncture to a weak CMC foliation of the ball.

A global application of the above theorem is that a compact, orientable Riemannian three-manifold not diffeomorphic to the three-sphere  $\mathbb{S}^3$  or to the real projective three-space  $\mathbb{P}^3$  does not admit any weak CMC foliation with a non-empty countable closed set of singularities; see [13] for this and other related results. In a different direction, it is natural to ask whether every closed, orientable three-manifold admits a Riemannian metric together with a smooth CMC

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<sup>4</sup>We mean here that  $M$  is allowed to intersect itself only in the way that the leaves of a weak  $H$ -lamination might intersect, see Definition 3.2.

foliation. The existence of such foliations follows from the next main theorem in [14] and the facts that every closed three-manifold has a differentiable structure and the Euler characteristic of any closed manifold of odd dimension is always zero; the proof of this theorem relies heavily on the seminal works of Thurston [26] on the existence of smooth codimension-one foliations of smooth closed  $n$ -manifolds and of Sullivan [25], which explains when such foliations are minimal with respect to some Riemannian metric.

**Theorem 1.4** *A smooth closed orientable  $n$ -manifold  $X$  admits a smooth CMC foliation  $\mathcal{F}$  for some Riemannian metric if and only if its Euler characteristic is zero. Furthermore,  $\mathcal{F}$  can be chosen so that the mean curvature function of its leaves changes sign.*

The paper is organized as follows. In Section 2 we extend to  $H$ -surfaces the Stability Lemma proved in [16] for the minimal case. In Section 3 we discuss some regularity aspects of  $H$  and CMC laminations, and define weak  $H$  and CMC laminations. Section 4 contains the proof of Theorem 1.3. Section 5 is devoted to prove the Local Removable Singularity Theorem 1.2 based on the previously proven minimal case (Theorem 1.1 in [16]). In Section 6 we demonstrate Theorem 1.1. Finally, in Section 7 we apply a rescaling argument and Theorem 1.1 to obtain Theorem 7.1 which describes the structure of any singular CMC foliation of a Riemannian three-manifold in a small neighborhood of any of its isolated singular points.

## 2 Stable surfaces with constant mean curvature which are complete outside of a point.

In [16] we extended the classical characterization of planes as the only orientable, complete, stable minimal surfaces in  $\mathbb{R}^3$  (do Carmo and Peng [7], Fischer-Colbrie and Schoen [8], Pogorelov [21]) to the case that completeness is only required outside a point, and called this result the *Stability Lemma*<sup>5</sup>. This result was a crucial tool in the proof of the minimal case of the Local Removable Singularity Theorem (Theorem 1.1 in [16]). Next we extend the Stability Lemma to the case of constant mean curvature surfaces, although this extension will not be directly used in the proof of Theorem 1.2: rather than this, the proof of Theorem 1.2 will be based on the validity of the minimal case of the Local Removable Singularity Theorem proven in [16], which in turn only uses the minimal case of the Stability Lemma. We remark that the notion of stability used in this paper is that the first eigenvalue of the stability operator for compactly supported variations is non-negative, in contrast with the usual stability notion related to the isoperimetric problem, where only compactly supported variations that preserve infinitesimally the volume are considered.

**Definition 2.1** An immersed surface  $M \subset \mathbb{R}^3 - \{\vec{0}\}$  is *complete outside the origin*, if every divergent path in  $M$  of finite length has as limit point the origin.

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<sup>5</sup> This minimal stability lemma was found independently by Colding and Minicozzi [3].

Let  $x: M \rightarrow \mathbb{R}^3$  be an isometric immersion of an orientable surface  $M$ . The mean curvature function  $H$  of  $x$  is constant with value  $c \in \mathbb{R}$  (in short,  $x(M)$  is an immersed  $c$ -surface) if and only if for all smooth compact subdomains of  $M$ ,  $x$  restricted to the subdomain is a critical point of the functional Area  $- 2c$  Volume. Given an  $H$ -surface  $x(M)$  and a function  $f \in C_0^\infty(M)$ , the second variation of Area  $- 2H$  Volume for any compactly supported variation of  $x$  whose normal component of the variational field is  $f$ , is well-known to be

$$Q(f, f) = - \int_M f L f \, dA,$$

where  $L = \Delta + |\sigma|^2 = \Delta - 2K + 4H^2$  is the *Jacobi operator* on  $M$  (here  $|\sigma|$  is the norm of the second fundamental form and  $K$  is the Gaussian curvature). The immersion is said to be *stable* if  $-L$  is a non-negative operator on  $M$ , i.e.,  $Q(f, f) \geq 0$  for every  $f \in C_0^\infty(M)$ . More generally, a Schrödinger operator  $-(\Delta + q)$  with  $q \in C^\infty(M)$  is called *non-negative* if

$$\int_M (|\nabla f|^2 - q f^2) \geq 0, \quad \text{for all } f \in C_0^\infty(M).$$

**Lemma 2.2 (Stability Lemma for  $H$ -surfaces)** *Let  $M \subset \mathbb{R}^3 - \{\vec{0}\}$  be a stable, immersed constant mean curvature (orientable if minimal) surface, which is complete outside the origin. Then, the closure  $\overline{M}$  of  $M$  is a plane.*

*Proof.* If  $\vec{0} \notin \overline{M}$ , then  $M$  is complete and so, it is a plane [7, 8, 21]. Assume now that  $\vec{0} \in \overline{M}$ . Consider the metric  $\tilde{g} = \frac{1}{R^2}g$  on  $M$ , where  $g$  is the metric induced by the usual inner product  $\langle, \rangle$  of  $\mathbb{R}^3$  and  $R$  is the distance to the origin in  $\mathbb{R}^3$ . Note that if  $M$  were a plane through  $\vec{0}$ , then  $\tilde{g}$  would be the metric on  $M - \{\vec{0}\}$  of an infinite cylinder of radius 1 with ends at  $\vec{0}$  and at infinity. Since  $(\mathbb{R}^3 - \{\vec{0}\}, \hat{g})$  with  $\hat{g} = \frac{1}{R^2}\langle, \rangle$ , is isometric to  $\mathbb{S}^2(1) \times \mathbb{R}$ , then  $(M, \tilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \hat{g})$  is complete.

The laplacians and Gauss curvatures of  $g, \tilde{g}$  are related by the equations  $\tilde{\Delta} = R^2\Delta$  and  $\tilde{K} = R^2(K + \Delta \log R)$ . Thus, the stability of  $(M, g)$  implies that the following operator is non-negative on  $M$ :

$$-\Delta + 2K - 4H^2 = -\frac{1}{R^2}(\tilde{\Delta} - 2\tilde{K} + q),$$

where  $q = 2R^2\Delta \log R + 4H^2R^2$ . Since  $\Delta \log R = \frac{2}{R^4}(R^2\langle p, \eta \rangle H + \langle p, \eta \rangle^2)$  where  $\eta$  is the unitary normal vector field to  $M$  (with respect to  $g$ ) for which  $H$  is the mean curvature, then

$$\frac{1}{4}q = H^2R^2 + \langle p, \eta \rangle H + \frac{\langle p, \eta \rangle^2}{R^2}. \quad (1)$$

Viewing the right-hand-side of (1) as a quadratic polynomial in the variable  $H$ , its discriminant is  $-3\langle p, \eta \rangle^2 \leq 0$ . Since the coefficient of  $H^2$  on the right-hand-side of (1) is  $R^2 \geq 0$ , we deduce that

$q \geq 0$  on  $M$ . Applying Theorem 2.9 in [17] to the operator  $\tilde{\Delta} - 2\tilde{K} + q$  on  $(M, \tilde{g})$ , we deduce that  $(M, \tilde{g})$  has at most quadratic area growth. This last property implies that every bounded solution of the equation  $\tilde{\Delta}u - 2\tilde{K}u + qu = 0$  has constant sign on  $M$  (see Theorem 1 in Manzano, Pérez and Rodríguez [11]).

Arguing by contradiction, suppose that  $(M, g)$  is not flat. Then, there exists a bounded Jacobi function  $u$  on  $(M, g)$  which changes sign (simply take a point  $p \in M$  and choose  $u$  as  $\langle \eta, a \rangle$  where  $a \in \mathbb{R}^3$  is a non-zero tangent vector to  $M$  at  $p$ ). Then clearly  $u$  satisfies  $\tilde{\Delta}u - 2\tilde{K}u + qu = 0$  on  $M$ . This contradiction proves the lemma.  $\square$

### 3 Weak $H$ -laminations and CMC laminations.

In this section we start by recalling the classical notion of lamination and discuss some previous results on regularity of these objects when the leaves have constant mean curvature. Then we will enlarge the class to admit *weak* laminations by allowing certain tangential intersections between the leaves. These weak laminations and foliations will be studied in subsequent sections.

**Definition 3.1** A *codimension-one lamination* of a Riemannian three-manifold  $N$  is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair  $(\mathcal{L}, \mathcal{A})$  satisfying:

1.  $\mathcal{L}$  is a closed subset of  $N$ ;
2.  $\mathcal{A} = \{\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta\}_\beta$  is an atlas of coordinate charts of  $N$  (here  $\mathbb{D}$  is the open unit disk in  $\mathbb{R}^2$ ,  $(0, 1)$  is the open unit interval in  $\mathbb{R}$  and  $U_\beta$  is an open subset of  $N$ ); note that although  $N$  is assumed to be smooth, we only require that the regularity of the atlas (i.e., that of its change of coordinates) is of class  $C^0$ ; in other words,  $\mathcal{A}$  is an atlas with respect to the topological structure of  $N$ .
3. For each  $\beta$ , there exists a closed subset  $C_\beta$  of  $(0, 1)$  such that  $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$ .

We will simply denote laminations by  $\mathcal{L}$ , omitting the charts  $\varphi_\beta$  in  $\mathcal{A}$  unless explicitly necessary. A lamination  $\mathcal{L}$  is said to be a *foliation* of  $N$  if  $\mathcal{L} = N$ . Every lamination  $\mathcal{L}$  decomposes into a collection of disjoint, connected topological surfaces (locally given by  $\varphi_\beta(\mathbb{D} \times \{t\})$ ,  $t \in C_\beta$ , with the notation above), called the *leaves* of  $\mathcal{L}$ . Note that if  $\Delta \subset \mathcal{L}$  is any collection of leaves of  $\mathcal{L}$ , then the closure of the union of these leaves has the structure of a lamination within  $\mathcal{L}$ , which we will call a *sublamination*.

A codimension-one lamination  $\mathcal{L}$  of  $N$  is said to be a *CMC lamination* if each of its leaves is smooth and has constant mean curvature (possibly varying from leaf to leaf). Given  $H \in \mathbb{R}$ , an  *$H$ -lamination* of  $N$  is a CMC lamination all whose leaves have the same mean curvature  $H$ . If  $H = 0$ , the  $H$ -lamination is called a *minimal lamination*.

Since the leaves of a lamination  $\mathcal{L}$  are disjoint, it makes sense to consider the second fundamental form  $\sigma_{\mathcal{L}}$  as being defined on the union of the leaves. A natural question to ask is whether or not the norm  $|\sigma_{\mathcal{L}}|$  of the second fundamental form of a (minimal,  $H$ - or CMC) lamination  $\mathcal{L}$  in a Riemannian three-manifold is locally bounded. Concerning this question, we make the following observations.

- O.1. If  $\mathcal{L}$  is a minimal lamination, then the 1-sided curvature estimates for minimal disks by Colding and Minicozzi [5, 6] imply that  $|\sigma_{\mathcal{L}}|$  is locally bounded (to prove this, one only needs to deal with limit leaves, where the 1-sided curvature estimates apply).
- O.2. As a consequence of recent work of Meeks and Tinaglia [19] on curvature estimates for embedded disks of positive constant mean curvature and a related 1-sided curvature estimate, a CMC lamination  $\mathcal{L}$  of a Riemannian three-manifold  $N$  has  $|\sigma_{\mathcal{L}}|$  locally bounded.

Given a sequence of CMC laminations  $\mathcal{L}_n$  of a Riemannian three-manifold  $N$  with uniformly bounded second fundamental form on compact subsets of  $N$ , a simple application of the uniform graph lemma for surfaces with constant mean curvature (see Colding and Minicozzi [4] or Pérez and Ros [20] from where this well-known result can be deduced) and of the Arzelà-Ascoli Theorem, gives that there exists a limit object of (a subsequence of) the  $\mathcal{L}_n$ , which in general fails to be a CMC lamination since two “leaves” of this limit object could intersect tangentially with mean curvature vectors pointing in opposite directions; nevertheless, if  $\mathcal{L}_n$  is a *minimal* lamination for every  $n$ , then the maximum principle ensures that the limit object is indeed a minimal lamination, see Proposition B1 in [5] for a proof. Still, in the general case of CMC laminations, such a limit object always satisfies the conditions in the next definition.

**Definition 3.2** A (codimension-one) *weak CMC lamination*  $\mathcal{L}$  of a Riemannian three-manifold  $N$  is a collection  $\{L_\alpha\}_{\alpha \in I}$  of (not necessarily injectively) immersed constant mean curvature surfaces called the *leaves* of  $\mathcal{L}$ , satisfying the following three properties.

1.  $\bigcup_{\alpha \in I} L_\alpha$  is a closed subset of  $N$ .
2. If  $p \in N$  is a point where either two leaves of  $\mathcal{L}$  intersect or a leaf of  $\mathcal{L}$  intersects itself, then each of these local surfaces at  $p$  lies at one side of the other (this cannot happen if both of the intersecting leaves have the same signed mean curvature as graphs over their common tangent space at  $p$ , by the maximum principle).
3. The function  $|\sigma_{\mathcal{L}}|: \mathcal{L} \rightarrow [0, \infty)$  given by

$$|\sigma_{\mathcal{L}}|(p) = \sup\{|\sigma_L|(p) \mid L \text{ is a leaf of } \mathcal{L} \text{ with } p \in L\}. \quad (2)$$

is uniformly bounded on compact sets of  $N$ .

Furthermore:



- If  $N = \bigcup_{\alpha} L_{\alpha}$ , then we call  $\mathcal{L}$  a *weak CMC foliation* of  $N$ .
- If the leaves of  $\mathcal{L}$  have the same constant mean curvature  $H$ , then we call  $\mathcal{L}$  a *weak  $H$ -lamination* of  $N$  (or  *$H$ -foliation*, if additionally  $N = \bigcup_{\alpha} L_{\alpha}$ ).

**Remark 3.3**

1. The function  $|\sigma_{\mathcal{L}}|$  defined in (2) for a CMC lamination is not necessarily continuous, as demonstrated by the following example: consider the union in  $\mathbb{R}^3$  of  $\Pi_0 = \{(x_1, x_2, x_3) \mid x_3 = 0\}$  and the sphere  $\mathbb{S}^2(p_1, 1)$  where  $p_1 = (0, 0, 1)$ . Also note that this example can be modified to create a weak CMC foliation  $\mathcal{F}$  of  $\mathbb{R}^3$  minus a point with non-continuous related function  $|\sigma_{\mathcal{F}}|$ : add to the previous example  $\mathcal{L}$  all planes  $\Pi_t = \{x_1, x_2, t\} \mid t < 0\}$ , and foliate the open set  $\{x_3 > 0\} - \{(0, 0, 1)\}$  by the spheres  $\mathbb{S}^2(p_t, t)$  with  $t \geq 1$  where  $p_t = (0, 0, t)$ , together with the spheres  $\mathbb{S}^2(p_1, r)$ ,  $r \in (0, 1)$ . In either case,  $|\sigma_{\mathcal{L}}|$  or  $|\sigma_{\mathcal{F}}|$  is not continuous at the origin.
2. A weak  $H$ -lamination for  $H = 0$  is a minimal lamination in the sense of Definition 3.1.
3. As a consequence of Observation O.2 above, every CMC lamination (resp. CMC foliation) of  $N$  is a weak CMC lamination (resp. weak CMC foliation).

The following proposition follows immediately from the definition of a weak  $H$ -lamination and the maximum principle for  $H$ -surfaces.

**Proposition 3.4** *Any weak  $H$ -lamination  $\mathcal{L}$  of a three-manifold  $N$  has a local  $H$ -lamination structure on the mean convex side of each leaf. More precisely, given a leaf  $L_{\alpha}$  of  $\mathcal{L}$  and given a small disk  $\Delta \subset L_{\alpha}$ , there exists an  $\varepsilon > 0$  such that if  $(q, t)$  denotes the normal coordinates for  $\exp_q(t\eta_q)$  (here  $\exp$  is the exponential map of  $N$  and  $\eta$  is the unit normal vector field to  $L_{\alpha}$  pointing to the mean convex side of  $L_{\alpha}$ ), then the exponential map  $\exp$  is an injective submersion in  $U(\Delta, \varepsilon) := \{(q, t) \mid q \in \text{Int}(\Delta), t \in (-\varepsilon, \varepsilon)\}$ , and the inverse image  $\exp^{-1}(\mathcal{L}) \cap \{q \in \text{Int}(\Delta), t \in [0, \varepsilon)\}$  is an  $H$ -lamination of  $U(\Delta, \varepsilon)$  in the pulled back metric, see Figure 2.*

**Definition 3.5** Let  $M$  be a complete, embedded surface in a Riemannian three-manifold  $N$ . A point  $p \in N$  is a *limit point* of  $M$  if there exists a sequence  $\{p_n\}_n \subset M$  which diverges to infinity in  $M$  with respect to the intrinsic Riemannian topology on  $M$  but converges in  $N$  to  $p$  as  $n \rightarrow \infty$ . Let  $\text{lim}(M)$  denote the set of all limit points of  $M$  in  $N$ ; we call this set the *limit set* of  $M$ . In particular,  $\text{lim}(M)$  is a closed subset of  $N$  and  $\overline{M} - M \subset \text{lim}(M)$ , where  $\overline{M}$  denotes the closure of  $M$ .

The above notion of limit point can be extended to the case of a lamination  $\mathcal{L}$  of  $N$  as follows: A point  $p \in \mathcal{L}$  is a *limit point* if there exists a coordinate chart  $\varphi_{\beta}: \mathbb{D} \times (0, 1) \rightarrow U_{\beta}$  as in Definition 3.1 such that  $p \in U_{\beta}$  and  $\varphi_{\beta}^{-1}(p) = (x, t)$  with  $t$  belonging to the accumulation set of  $C_{\beta}$ . The notion of limit point can be also extended to the case of a weak  $H$ -lamination of  $N$ ,

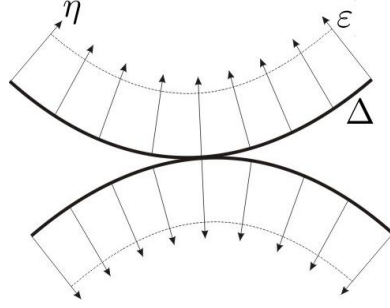


Figure 2: The leaves of a weak  $H$ -lamination with  $H \neq 0$  can intersect each other or themselves, but only tangentially with opposite mean curvature vectors. Nevertheless, on the mean convex side of these locally intersecting leaves, there is a lamination structure.

by using that such an weak  $H$ -lamination has a local lamination structure at the mean convex side of any of its points, given by Proposition 3.4. It is easy to show that if  $p$  is a limit point of a lamination  $\mathcal{L}$  (resp. of a weak  $H$ -lamination), then the leaf  $L$  of  $\mathcal{L}$  passing through  $p$  consists entirely of limit points of  $\mathcal{L}$ ; in this case,  $L$  is called a *limit leaf* of  $\mathcal{L}$ .

## 4 Proof of Theorem 1.3.

We now prove the universal curvature estimates for weak CMC foliations stated in Theorem 1.3. Let  $N$  be a compact Riemannian three-manifold possibly with boundary, whose absolute sectional curvature is at most  $\Lambda \geq 0$ . Let  $\mathcal{F}$  be a weak CMC foliation of  $N$  and  $p \in \text{Int}(N)$ . Recall that  $|\sigma_{\mathcal{F}}|$  is the function defined in (2).

Note that  $|\sigma_{\mathcal{F}}|(p) \min\{\text{dist}_N(p, \partial N), \frac{\pi}{\sqrt{\Lambda}}\}$  is invariant under rescaling of the ambient metric. This invariance implies that we can fix  $\Lambda > 0$  and prove that  $|\sigma_{\mathcal{F}}|(p) \min\{\text{dist}_N(p, \partial N), \frac{\pi}{\sqrt{\Lambda}}\}$  is bounded independently of the compact Riemannian three-manifold  $N$  with boundary whose absolute sectional curvature is at most  $\Lambda$  and independently of the weak CMC foliation  $\mathcal{F}$  of  $N$ .

We fix  $\Lambda > 0$ . Arguing by contradiction, assume that there exists a sequence of weak CMC foliations  $\mathcal{F}_n$  of compact three-manifolds  $N_n$  with boundary, such that the absolute sectional curvature of  $N_n$  is at most  $\Lambda$ , and there exists a sequence of points  $p_n$  in leaves  $L_n$  of  $\mathcal{F}_n$  with  $|\sigma_{L_n}|(p_n) \min\{r_n, \frac{\pi}{\sqrt{\Lambda}}\} \geq n$  for all  $n$ , where  $r_n = d_{N_n}(p_n, \partial N_n)$ . After replacing  $r_n$  by  $\min\{r_n, \frac{\pi}{\sqrt{\Lambda}}\}$  and  $N_n$  by the closed metric ball  $\overline{B}_{N_n}(p_n, r_n)$ , we can assume  $r_n \leq \frac{\pi}{\sqrt{\Lambda}}$ .

Since the sectional curvature of  $N_n$  is at most  $\Lambda$ , a standard comparison argument for zeros of Jacobi fields on geodesics gives that the exponential map

$$\exp_{p_n} : B(\vec{0}, r_n) \subset T_{p_n} N_n \rightarrow B_{N_n}(p_n, r_n)$$

is a local diffeomorphism from the ball of radius  $r_n$  centered at the origin in the tangent space to  $N_n$  at  $p_n$  (endowed with the ambient metric at  $p_n$ ) onto the metric ball  $B_{N_n}(p_n, r_n)$ . After lifting the ambient metric of  $N_n$  to  $B(\vec{0}, r_n)$ , we can consider the above map to be a local isometry (note that the metric on  $B(\vec{0}, r_n)$  depends on  $n$ ). We can also consider the weak CMC foliation  $\mathcal{F}_n$  to be a weak CMC foliation of  $B(\vec{0}, r_n)$  with the pulled back metric. Next consider the homothetic expansion of the metric in  $B(\vec{0}, r_n)$  centered at the origin with ratio  $\frac{\pi}{r_n \sqrt{\Lambda}}$ . After this new normalization, we have the following properties:

- (P1)  $B(\vec{0}, r_n)$  becomes a metric ball of radius  $\frac{\pi}{\sqrt{\Lambda}}$ , which we will denote by  $B_n(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})$ , since its Riemannian metric still depends on  $n$ .
- (P2) The absolute sectional curvature of  $B_n(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})$  is less than or equal to  $\Lambda$ .
- (P3) We have a related CMC foliation  $\mathcal{F}'_n$  on  $B_n(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})$  so that the second fundamental form of some leaf  $L'_n$  of  $\mathcal{F}'_n$  passing through  $\vec{0}$  satisfies  $|\sigma_{L'_n}|(\vec{0}) \frac{\pi}{\sqrt{\Lambda}} \geq n$  for all  $n \in \mathbb{N}$ .

By Lemma 2.2 in [23], the injectivity radius function  $I_n$  of  $B_n(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})$  satisfies  $I_n(x) \geq \frac{\pi}{4\sqrt{\Lambda}}$  for all points  $x \in B_n(\vec{0}, \frac{\pi}{4\sqrt{\Lambda}})$ . By Theorem 2.1 in [23], given  $\alpha \in (0, 1)$  there exists  $r_0 > 0$  (only depending on  $\Lambda$  but not on  $n$ ) so that we can pick harmonic coordinates in the metric ball  $B_n(x, r_0)$  centered at any point  $x \in B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$  of radius  $r_0$ . For the notion of harmonic coordinates, see [23]; the only property we will use here about these harmonic coordinates is that the metric tensor on  $B_n(x, r_0)$  (which depends on  $n$ ) is  $C^{1,\alpha}$ -controlled.

Fix  $n \in \mathbb{N}$  and let  $q_n \in B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$  be a supremum of the function

$$q \in B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}}) \mapsto f_n(q) = |\sigma_{\mathcal{F}'_n}|(q) d_n(q, \partial B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})), \quad (3)$$

where given  $q \in B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$ ,  $|\sigma_{\mathcal{F}'_n}|(q) = \sup\{|\sigma_{L'}|(q) \mid L' \in \mathcal{F}'_n, q \in L'\}$  and  $d_n$  denotes extrinsic distance in  $B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$ . We observe that the following properties hold:

- The supremum in (3) exists since the second fundamental form of all leaves of  $\mathcal{F}'_n$  is uniformly bounded in  $B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$  by definition of weak CMC foliation.
- $f_n$  may not be continuous (as  $q \mapsto |\sigma_{\mathcal{F}'_n}|(q)$  might fail to be continuous) but still it is bounded, and it vanishes at  $\partial B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$ . A similar argument as in the observation above shows that the supremum of  $f_n$  is attained at an interior point of  $B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}})$ .
- The value of  $f_n$  at  $\vec{0}$  tends to  $\infty$  as  $n \rightarrow \infty$ .

Let  $s_n = \min\{r_0, \frac{1}{2}d_n(q_n, \partial B_n(\vec{0}, \frac{\pi}{8\sqrt{\Lambda}}))\}$  and let  $\lambda_n = |\sigma_{\mathcal{F}'_n}|(q_n)$ . After rescaling the metric of the ball  $B_n(q_n, s_n)$  of radius  $s_n$  centered at  $q_n$  by the factor  $l_n$ , we have associated weak CMC

foliations  $\mathcal{F}_n''$  of  $\lambda_n B_n(q_n, s_n)$  such that the norm of the second fundamental form of  $\mathcal{F}_n''$  is at most 2 everywhere and is equal to 1 at the center  $q_n$  of this ball.

Using the techniques described in [23], it follows that a subsequence of the  $\mathcal{F}_n''$  converges to a weak CMC foliation  $\mathcal{Z}$  of  $\mathbb{R}^3$ ; next we sketch an explanation of these techniques: First, one uses the above harmonic coordinates to show that the coordinatized Riemannian manifolds  $l_n B_n(q_n, s_n)$  converge uniformly on compact subsets of  $\mathbb{R}^3$  in the  $C^{1,\alpha}$ -Euclidean topology to  $\mathbb{R}^3$  endowed with its Euclidean metric. Then one shows that the leaves in  $\mathcal{F}_n''$  can be locally written as graphs of functions defined over Euclidean disks of uniformly controlled size (this property follows from the uniform graph lemma for surfaces with constant mean curvature, note that for this we need an uniform bound on the second fundamental form of the leaves of  $\mathcal{F}_n''$ , which is obtained by the blow-up process). Another consequence of the uniform graph lemma is that one obtains uniform local  $C^2$ -bounds for the graphing functions. The next step consists of using that the graphing functions satisfy the corresponding mean curvature equation, which is an elliptic PDE of second order whose coefficients have uniform  $C^{0,\alpha}$ -estimates (this follows from the  $C^2$ -bounds for the graphing functions and the  $C^{1,\alpha}$ -control of the ambient metric on  $l_n B_n(q_n, s_n)$ , see Lemma 2.4 in [23]), together with Schauder estimates to conclude local uniform  $C^{2,\alpha}$ -bounds for the graphing functions of the leaves of  $\mathcal{F}_n''$ . Finally, these local uniform  $C^{2,\alpha}$ -bounds for the graphing functions allow us to use the Arzelà-Ascoli Theorem to obtain convergence (after extracting a subsequence) in the  $C^2$ -topology to limit graphing functions of class  $C^2$ . Using that the convergence is  $C^2$  one can pass to the limit the mean curvature equations satisfied by the graphing functions in the sequence, thereby producing a (local) limit weak CMC foliation of an open set of  $\mathbb{R}^3$ . Finally, a diagonal argument produces a global limit weak CMC foliation  $\mathcal{Z}$  of  $\mathbb{R}^3$  of a subsequence of the  $\mathcal{F}_n''$ . For details, see [23].

The above process insures that the limit weak CMC foliation  $\mathcal{Z}$  of  $\mathbb{R}^3$  satisfies the following properties:

1. The second fundamental form of the leaves of  $\mathcal{Z}$  is bounded in absolute value by 1 (in particular, there is a bound on the mean curvature of every leaf of  $\mathcal{Z}$ ), and there is a leaf  $\Sigma$  of  $\mathcal{Z}$  passing through the origin which is not flat.
2.  $\mathcal{Z}$  is not a minimal foliation (otherwise  $\mathcal{Z}$  would consist entirely of planes, contradicting the existence of  $\Sigma$ ).

Since the leaves of  $\mathcal{Z}$  have uniformly bounded second fundamental forms, after a sequence of translations of  $\mathcal{Z}$  in  $\mathbb{R}^3$ , we obtain another limit weak CMC foliation  $\widehat{\mathcal{Z}}$  of  $\mathbb{R}^3$  with a leaf  $\widehat{L}$  passing through the origin which has non-zero *maximal* mean curvature among all leaves of  $\widehat{\mathcal{Z}}$ . But the two-sided surface  $\widehat{L}$  is then stable by Proposition 5.4 in [17] and since it is also complete, then  $\widehat{L}$  must be flat (see for instance Lemma 2.2 for a proof of this well-known result). This contradiction finishes the proof of the theorem.  $\square$

In Section 6 we will use the next corollary, which follows immediately from Theorem 1.3 after scaling the estimate for balls of radius 1.

**Corollary 4.1** *There exists an  $A > 0$  such that if  $\mathcal{F}$  is a weak CMC foliation of a ball  $\mathbb{B}(p, R) \subset \mathbb{R}^3$ , then  $|\sigma_{\mathcal{F}}|(p) \leq A/R$ , where  $|\sigma_{\mathcal{F}}|$  is given by (2).*

## 5 Proof of Theorem 1.2.

Let  $\mathcal{L}$  be a weak  $H$ -lamination of a punctured ball  $B_N(p, r)$  in a Riemannian three-manifold  $N$ , such that  $|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C$  for some  $C > 0$ . To prove Theorem 1.2, it suffices to check that  $\mathcal{L}$  extends to a weak  $H$ -lamination of  $B_N(p, r)$  for a smaller  $r > 0$ . Throughout this section, we will assume without loss of generality that  $r$  is sufficiently small so that the exponential map  $\exp_p$  restricted to  $B(\vec{0}, r) \subset T_p N = \mathbb{R}^3$  induces  $\mathbb{R}^3$ -coordinates on  $B_N(p, r)$ .

**Lemma 5.1** *Let  $\mathcal{L}$  be a weak  $H$ -lamination of a punctured ball  $B_N(p, r)$  in a Riemannian three-manifold  $N$ , such that  $|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C$  for some  $C > 0$ . Then for every sequence of positive numbers  $C_n \searrow 0$ , there exists another sequence  $r_n \searrow 0$  such that*

$$|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C_n \quad \text{in } \mathcal{L} \cap B_N(p, r_n).$$

*Proof.* Arguing by contradiction, suppose that the lemma fails. It follows that there exists an  $\varepsilon > 0$  and a sequence of points  $p_n \in \mathcal{L}$  such that  $\lim_{n \rightarrow \infty} p_n = p$  and  $\varepsilon \leq |\sigma_{\mathcal{L}}|(p_n) d_N(p, p_n)$  for each  $n$ . Let  $\lambda_n = \frac{1}{d_N(p, p_n)}$  and consider the sequence of rescaled weak  $\frac{H}{\lambda_n}$ -laminations  $\mathcal{L}_n = \lambda_n \mathcal{L} \subset \lambda_n B_N(p, r)$ ; here by  $\lambda_n B_N(p, r)$  we mean  $B_N(p, r)$  endowed with the Riemannian metric  $l_n^2 \langle \cdot, \cdot \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the metric on  $N$ .

As  $H$  is fixed,  $l_n \rightarrow \infty$ ,  $|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C$  and  $|\sigma_{\mathcal{L}}| d_N(p, \cdot)$  is invariant under homothetic rescalings of the metric around  $p$ , then there exists a subsequence of the weak  $\frac{H}{l_n}$ -laminations  $\mathcal{L}_n$  that converges to a minimal lamination  $\mathcal{L}'$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , which furthermore satisfies  $|\sigma_{\mathcal{L}'}| R \leq C$  in  $\mathbb{R}^3 - \{\vec{0}\}$  (recall that  $R$  stands for radial distance in  $\mathbb{R}^3$  to the origin). As  $\varepsilon \leq |\sigma_{\mathcal{L}}|(p_n) d_N(p, p_n)$  for all  $n$ , then  $\varepsilon \leq |\sigma_{\mathcal{L}'}|(q_\infty)$  for some  $q_\infty \in \mathcal{L}' \cap \mathbb{S}^2(1)$ ; thus the lamination  $\mathcal{L}'$  is not flat. In this setting, Corollary 6.3 in [16] gives that the closure  $\overline{\mathcal{L}'}$  of  $\mathcal{L}'$  in  $\mathbb{R}^3$  consists of a single leaf  $\overline{L'}$ , which is a non-flat minimal surface with finite total curvature (clearly  $\mathcal{L}' = \{L'\}$  where  $L' = \overline{L'} - \{\vec{0}\}$ ).

**Claim A:**  $\overline{L'}$  contains the origin  $\vec{0}$ .

To see this, it suffices to show that the distance sphere  $S_N^2(p, \delta)$  intersects  $\mathcal{L}$  for every  $\delta > 0$  sufficiently small. Otherwise, there exists  $\delta_1 > 0$  such that the following properties hold:

1.  $S_N^2(p, \delta_1) \cap \mathcal{L} = \emptyset$ .
2. For all  $\delta \in (0, \delta_1]$ , the mean curvature function of  $S_N^2(p, \delta)$  is strictly greater than  $H$ .
3. The family  $\{S_N^2(p, \delta) \mid \delta \in (0, \delta_1]\}$  foliates the punctured closed ball  $\overline{B}_N(p, \delta_1) - \{p\}$ .

Since  $B_N(p, \delta_1)$  intersects  $\mathcal{L}$  and  $\mathcal{L}$  is a closed subset of  $\overline{B}_N(p, \delta_1) - \{p\}$ , then there exists a largest  $\delta_2 \in (0, \delta_1)$  such that  $S_N^2(p, \delta_2) \cap \mathcal{L} \neq \emptyset$ . This contradicts the mean curvature comparison principle, which proves Claim A.

**Claim B:** *There exists  $r' \in (0, r)$  such that the following properties hold:*

- (B1) *The intersection of  $\mathcal{L}$  with  $\overline{B}_N(p, r')$  consists of a single leaf  $L$  of the induced lamination. Furthermore,  $L \cap S_N^2(p, r')$  is a simple closed curve along which  $L$  and  $S_N^2(p, r')$  intersect almost orthogonally.*
- (B2)  *$L \cap \overline{B}_N(p, r')$  is properly embedded in  $\overline{B}_N(p, r') - \{p\}$ , with  $p$  in its closure.*

To see that Claim B holds, first note that by Claim A, the intersection of  $L'$  with any closed ball  $\overline{\mathbb{B}}(R)$  centered at  $\vec{0}$  of sufficiently small radius  $R > 0$  is a punctured disk which is almost orthogonal to  $\mathbb{S}^2(R)$ . Since  $L'$  is not flat, then the convergence of the laminations  $\mathcal{L}_n$  to  $L'$  has multiplicity one (by Lemma 4.2 in [15]). Thus, we deduce that for  $n$  large and  $r'_n := d_N(p_n, p)R$ , there exists a unique leaf  $L(n)$  of  $\mathcal{L}$  that intersects  $S_N^2(p, r'_n)$ , and this intersection is a simple closed curve along which  $L(n)$  and  $S_N^2(p, r'_n)$  intersect almost orthogonally. If  $L(n)$  were not the unique leaf of  $\mathcal{L}$  that intersects  $B_N(p, r'_n)$ , then  $\mathcal{L} \cap B_N(p, r'_n)$  would contain a non-empty sublamination which does not intersect  $S_N^2(p, r'_n)$ . A similar comparison argument for the mean curvature as in the proof of Claim A shows that this is impossible for  $n$  sufficiently large. Hence item (B1) above holds by taking  $r' = r'_n$  and  $L = L(n)$  for  $n$  large.

Suppose that  $L$  is a limit leaf of  $\mathcal{L}$ . In this case, item (B1) implies that every point in  $L \cap S_N^2(p, r')$  is the limit in  $N$  of a sequence of points of  $L$  itself. This is impossible, since  $L$  and  $S_N^2(p, r')$  intersect almost orthogonally in a Jordan curve. Thus,  $L$  is not a limit leaf of  $\mathcal{L}$ . We next prove item (B2): If  $L$  were not proper in  $B_N(p, r') - \{p\}$ , then  $\mathcal{L} \cap B_N(p, r')$  would contain a limit leaf, which therefore would not be  $L$ ; this contradicts (B1). Finally, if  $p$  is not in the closure of  $L$ , then  $p$  is at positive distance from  $L$ . This contradicts (B1) together with  $\varepsilon \leq |\sigma_{\mathcal{L}}|(p_n)d(p, p_n)$  as  $p_n$  converges to  $p$ . Now (B2) is proved, as well as Claim B.

We next finish the proof of the lemma. Since  $L$  is properly embedded in  $B_N(p, r') - \{p\}$ , then  $L$  is a locally rectifiable current in  $B_N(p, r') - \{p\}$ . As  $L$  has bounded mean curvature (actually constant), then Theorem 3.1 in Harvey and Lawson [10] implies that  $L$  has finite area. Since  $L$  has bounded mean curvature and finite area, the monotonicity formula in Corollary 5.3 of Allard [2] implies that  $L$  has a well-defined finite density at  $p$ . In this setting, we can apply Theorem 6.5 in [2] to deduce that under any sequence of homothetic expansions  $\{L'_n\}_n$  of  $L$ , the surfaces  $L'_n$  converge (up to a subsequence) to a cone  $\mathcal{C}_L \subset \mathbb{R}^3$  (depending on the sequence), which is the cone over a stationary, integral one-dimensional varifold  $\Gamma$  in the unit two-sphere of  $\mathbb{R}^3$ , and  $\mathcal{C}_L$  is flat at its smooth points. But the blow-up limit  $L'$  is smooth and not flat, which is a contradiction. This contradiction proves the lemma.  $\square$

By Lemma 5.1, it follows that the following property holds:

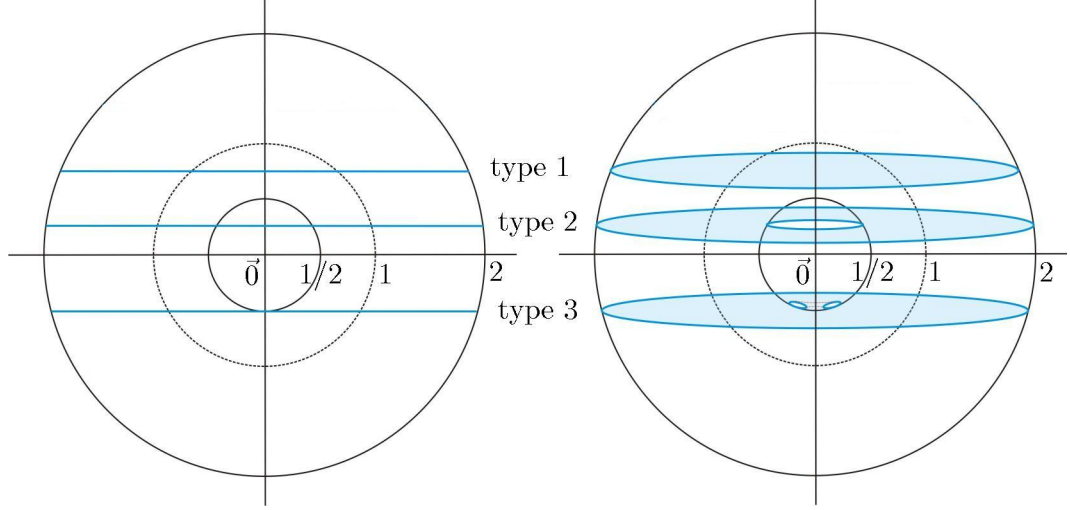


Figure 3: Type 1, 2, 3 connected components of  $\mathcal{L}_\varepsilon$ .

**(P)** Under rescaling by every sequence  $\{\lambda_n\}_n \subset (0, \infty)$  with  $\lambda_n \rightarrow \infty$ , a subsequence of the weak  $\frac{H}{l_n}$ -laminations  $\mathcal{L}_n = \lambda_n \mathcal{L} \subset \lambda_n B_N(p, r)$  converges in  $\mathbb{R}^3 - \{\vec{0}\}$  to a lamination  $\mathcal{L}'$  of  $\mathbb{R}^3$  by parallel planes. (Note that  $\mathcal{L}'$  might depend on  $\{l_n\}_n$ ).

**Proposition 5.2** *Theorem 1.2 holds in the particular case  $N = \mathbb{R}^3$  and  $p = \vec{0}$ .*

*Proof.* As property **(P)** holds, it follows that for  $\varepsilon > 0$  sufficiently small, in the annular domain  $A = \{x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2\}$ , the normal vectors to the leaves of  $(\frac{1}{\varepsilon} \mathcal{L}) \cap A$  are almost parallel, and after a rotation (which might depend on  $\varepsilon$ ), we will assume that the unit normal vector to the leaves of  $(\frac{1}{\varepsilon} \mathcal{L}) \cap A$  lies in a small neighborhood of  $\{\pm(0, 0, 1)\}$ . Hence, for such a sufficiently small  $\varepsilon$ , each component  $C$  of  $(\frac{1}{\varepsilon} \mathcal{L}) \cap A$  that intersects  $\mathbb{S}^2(1)$  is of one of the following four types, see Figure 3:

Type 1.  $C$  is a compact disk with boundary  $\Gamma(C)$  in  $\mathbb{S}^2(2)$ .

Type 2.  $C$  is a compact annulus with one boundary curve  $\Gamma(C)$  in  $\mathbb{S}^2(2)$  and the other boundary curve in  $\mathbb{S}^2(\frac{1}{2})$ .

Type 3.  $C$  is a compact planar domain whose boundary consists of a single closed curve  $\Gamma(C)$  in  $\mathbb{S}^2(2)$  together with at least two closed curves in  $\mathbb{S}^2(\frac{1}{2})$ , and where  $\Gamma(C)$  bounds a compact disk in  $\frac{1}{\varepsilon} \mathcal{L}$ ;

Type 4.  $C$  is an infinite multigraph whose limit set consists of two compact components of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  of type 2 (any such spiraling component does not intersect the intersection of  $A$  with an open slab of small width around height  $\pm\frac{1}{2}$ ).

We also define  $A(n) = \{x \in \mathbb{R}^3 \mid \frac{1}{2^{2n+1}} \leq |x| \leq \frac{1}{2^{2n-1}}\}$  for each  $n \in \mathbb{N} \cup \{0\}$  (so  $A = A(0)$ ). Note that  $\bigcup_{n \in \mathbb{N} \cup \{0\}} A(n) = \overline{\mathbb{B}}(2) - \{\vec{0}\}$ . Given a component  $C$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$ , let  $\Delta_C$  be the leaf of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap [\overline{\mathbb{B}}(2) - \{\vec{0}\}]$  that contains  $C$ . Given  $n \in \mathbb{N} \cup \{0\}$  fixed, the above division of components  $C$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A(0)$  can be directly adapted to components  $\Delta_C(n) = \Delta_C \cap A(n)$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A(n)$ . We make the following elementary observations:

- (O1) If for some  $n \in \mathbb{N} \cup \{0\}$ ,  $\Delta_C(n)$  is of type 4, then  $\Delta_C(n')$  is of type 4 for every  $n' \in \mathbb{N} \cup \{0\}$ , and  $\Delta_C$  has  $\vec{0}$  in its closure.
- (O2) If for some  $n \in \mathbb{N} \cup \{0\}$ ,  $\Delta_C(n)$  is either empty, of type 1 or of type 3, then  $\Delta_C$  is a disk which is at positive distance from  $\vec{0}$ .
- (O3) If neither (O1) nor (O2) occur, then  $\Delta_C(n)$  is of type 2 for every  $n$  and thus,  $\Delta_C$  is a proper annulus limiting to  $\vec{0}$ .

**Assertion 5.3** *There exists a leaf of  $\frac{1}{\varepsilon}\mathcal{L}$  having  $\vec{0}$  in its closure.*

*Proof.* If  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  contains a component of type 4, then (O1) implies that our claim holds. If  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  contains a component  $C$  of type 2 such that  $\Delta_C(n)$  is of type 2 for every  $n$ , then (O3) insures that  $\Delta_C$  contains  $\vec{0}$  in its closure and we are also done. We will prove that the remaining case is impossible and this will finish the proof of this assertion.

The remaining case is that for every component  $C$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$ , there exists  $n \in \mathbb{N} \cup \{0\}$  such that  $\Delta_C(n)$  is empty, of type 1 or of type 3; in the case that  $\Delta_C(n)$  is of type 3, then  $\Delta_C(n')$  is empty for every  $n' > n + 1$ . By (O2), we have that  $(\frac{1}{\varepsilon}\mathcal{L}) \cap [\overline{\mathbb{B}}(2) - \{\vec{0}\}]$  consists of an (infinite) collection of pairwise disjoint compact disks. Since  $\vec{0}$  is in the closure of  $\mathcal{L}$ , there exists a sequence of points  $\{p_m\}_{m \in \mathbb{N}}$  in compact disk leaves  $D(p_m)$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap [\overline{\mathbb{B}}(2) - \{\vec{0}\}]$ , such that the  $p_m$  converge to  $\vec{0}$  as  $m \rightarrow \infty$ . We define for every  $k \in \mathbb{N}$ ,

$$\mathcal{D}(k) = \bigcup_{m=k}^{\infty} D(p_m).$$

We claim that there exists  $k \in \mathbb{N}$  such that  $\mathcal{D}(k)$  is not closed in  $\overline{\mathbb{B}}(2) - \{\vec{0}\}$ . Otherwise,  $\{\mathcal{D}(k) \cap A \mid k \in \mathbb{N}\}$  is a collection of closed subsets of the compact space  $A$ , which clearly satisfies the finite intersection property; therefore, there exists a point  $q \in [\bigcap_{k=1}^{\infty} \mathcal{D}(k)] \cap A$ . In particular,  $q \in D(p_j)$  for some  $j \in \mathbb{N}$ . But as  $D(p_j)$  is disjoint from  $\mathcal{D}(j+1)$ , we arrive to a contradiction. This contradiction proves that there exists  $k \in \mathbb{N}$  such that  $\mathcal{D}(k)$  is not closed in  $\overline{\mathbb{B}}(2) - \{\vec{0}\}$ .



Since  $\mathcal{D}(k)$  is not closed in  $\overline{\mathbb{B}}(2) - \{\vec{0}\}$ , then there exists a point  $x \in \overline{\mathbb{B}}(2) - \{\vec{0}\}$  which is in the closure of  $\mathcal{D}(k)$  but not in  $\mathcal{D}(k)$ . As  $\frac{1}{\varepsilon}\mathcal{L}$  is closed in  $\overline{\mathbb{B}}(2) - \{\vec{0}\}$  and  $\mathcal{D}(k) \subset \frac{1}{\varepsilon}\mathcal{L}$ , then  $x \in \frac{1}{\varepsilon}\mathcal{L}$ . Thus, there exists a leaf component  $D_x$  of  $\frac{1}{\varepsilon}\mathcal{L}$  passing through  $x$ , which is disjoint from  $\mathcal{D}(k)$  as  $x \in D_x - \mathcal{D}(k)$ . Furthermore, by our previous arguments,  $D_x$  is topologically a closed disk which is at positive distance from  $\vec{0}$ . Choose a compact neighborhood  $U$  of  $D_x$  in  $\overline{\mathbb{B}}(2) - \{\vec{0}\}$  which does not contain the origin. Since  $\frac{1}{\varepsilon}\mathcal{L}$  is a lamination and  $D_x$  is a closed disk leaf,  $U$  can be chosen so that every leaf of  $\frac{1}{\varepsilon}\mathcal{L}$  which intersects  $U$  is entirely contained in  $U$ . It follows that there exists a subsequence of the disks  $\{D(p_m)\}_m$  which is contained in  $U$ . This is clearly a contradiction, as the  $D(p_m)$  contain points that converge to  $\vec{0}$ . This proves Assertion 5.3.  $\square$

**Assertion 5.4** *There exists a leaf  $L$  of  $\frac{1}{\varepsilon}\mathcal{L}$  which is a proper annulus having  $\vec{0}$  in its closure. Moreover, every such  $L$  extends smoothly across  $\vec{0}$ .*

*Proof.* By Assertion 5.3, there exists a component  $C$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  such that  $\Delta_C$  has  $\vec{0}$  in its closure. By observation (O2) above,  $\Delta_C(n)$  is of type 2 or 4 for all  $n \in \mathbb{N}$ . If  $\Delta_C(n)$  is of type 2 for some  $n$  (hence for all  $n$ ), then  $\Delta_C$  is a proper annulus limiting to  $\vec{0}$ . Otherwise,  $\Delta_C(n)$  is of type 4 for some  $n$  (hence for all  $n$ ), and thus the limit set of  $C$  in  $A$  produces two compact components  $C_1, C_2$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  each of which is of type 2 and such that  $\Delta_{C_1}, \Delta_{C_2}$  are proper annuli limiting to  $\vec{0}$ . This proves the first sentence of the assertion.

We next choose a proper annular leaf  $L$  of  $\frac{1}{\varepsilon}\mathcal{L}$  with  $\vec{0} \in \overline{L}$  and check that  $L$  extends smoothly across  $\vec{0}$ . Since every blow-up limit of  $L$  is a lamination of  $\mathbb{R}^3$  by parallel planes (property (P)), then  $L$  intersects small spheres  $\mathbb{S}^2(r')$  of radius  $0 < r' \ll r$  almost orthogonally in a curve of length no greater than  $3\pi r'$ , and thus,  $L$  has finite area. The conformal structure of  $L$  must be the one of a punctured disk, as follows from the fact that under the conformal change of metric  $\tilde{g} = \frac{1}{R^2}\langle, \rangle$ ,  $(L, \tilde{g}|_L)$  has linear area growth (here,  $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$  and  $\langle, \rangle$  is the inner product in  $\mathbb{R}^3$ , recall that our present goal is to prove Proposition 5.2, which is the  $\mathbb{R}^3$ -case of Theorem 1.2). Since  $L$  has finite area and is conformally a punctured disk, then  $L$  can be conformally parameterized by a mapping from a punctured disk into  $\mathbb{R}^3$  with finite energy. In this setting, the main theorem in [9] (which holds true even if we exchange our current ambient manifold  $\mathbb{R}^3$  by any Riemannian three-manifold whose sectional curvature is bounded from above and whose injectivity radius is bounded away from zero, conditions which will be satisfied in the general setting for  $N$  that will be dealt with in Proposition 5.7 below, since we work in an arbitrarily small ball  $B_N(p, r')$ ,  $r' \in (0, r)$ ), implies that  $L$  extends  $C^1$  through  $p$  and so, standard elliptic theory gives that  $L$  extends smoothly across  $\vec{0}$  as a mapping. Since  $L$  is embedded around  $\vec{0}$ , then the extended image surface is also smooth. This completes the proof of Assertion 5.4.  $\square$

**Assertion 5.5** *There are no type 4 components of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$ .*

*Proof.* Arguing by contradiction, if there exists a type 4 component  $C$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$ , then the limit set of  $C$  in  $A$  produces two compact components  $C_1, C_2$  of  $(\frac{1}{\varepsilon}\mathcal{L}) \cap A$  each of which is of type 2 and such that  $\Delta_{C_1}, \Delta_{C_2}$  are proper annuli limiting to  $\vec{0}$ . By Assertion 5.4, both  $\Delta_{C_1}, \Delta_{C_2}$  extend smoothly through  $\vec{0}$  by the previous paragraph. This contradicts the usual maximum principle for  $H$ -surfaces, as both  $\Delta_{C_1}, \Delta_{C_2}$  have the same orientation at  $\vec{0}$  since the orientation of the multigraph component  $C$  of type 4 induces the orientation of both  $\Delta_{C_1}, \Delta_{C_2}$ .  $\square$

By Observation **(O2)** and Assertions 5.4 and 5.5, every leaf of  $\mathcal{L}$  which limits to  $\vec{0}$  is a proper annulus which extends smoothly across  $\vec{0}$  (hence by the maximum principle there are at most two of them, with common tangent plane  $\Pi$  at  $\vec{0}$  and oppositely pointing mean curvature vectors) and there exists at least one such proper annulus. Therefore, property **(P)** can now be improved to the following property:

**(P)'** *Under rescaling by every sequence  $\{\lambda_n\}_n \subset (0, \infty)$  with  $\lambda_n \rightarrow \infty$ , a subsequence of the weak  $\frac{H}{l_n}$ -laminations  $\mathcal{L}_n = \lambda_n \mathcal{L} \subset \lambda_n B_N(p, r)$  converges in  $\mathbb{R}^3 - \{\vec{0}\}$  to a lamination  $\mathcal{L}'$  of  $\mathbb{R}^3$  by planes parallel to  $\Pi$ .*

Let  $F$  be one of the at most two proper annular leaves in  $\mathcal{L}$  limiting to  $\vec{0}$ . Let  $\bar{F}$  be the extended  $H$ -disk obtained after attaching the origin to  $F$ . Consider an intrinsic geodesic disk  $D_{\bar{F}}(\vec{0}, \delta)$  in  $\bar{F}$  centered at  $\vec{0}$  with radius  $\delta$ , and let  $\eta$  be the unit normal vector field to  $D_{\bar{F}}(\vec{0}, \delta)$ . Pick coordinates  $q = (x, y)$  in  $D_{\bar{F}}(\vec{0}, \delta)$  and let  $t \in [-\tau, \tau] \mapsto \gamma_q(t) = q + t\eta(q)$  be the straight line in  $\mathbb{R}^3$  passing through  $q$  with velocity vector  $\eta(q)$  (here  $\tau > 0$  is small and independent of  $q \in D_{\bar{F}}(\vec{0}, \delta)$  so that the straight lines  $\gamma_q$  do not intersect each other). Then for some  $\tau > 0$  small,  $(x, y, t)$  produces “cylindrical” normal coordinates in a neighborhood  $V$  of  $\vec{0}$  in  $\mathbb{R}^3$ , and we can consider the natural projection

$$\Phi: V \rightarrow D_{\bar{F}}(\vec{0}, \delta), \quad \Phi(x, y, t) = (x, y).$$

Since by **(P)'** every blow-up limit of  $\mathcal{L}$  from  $\vec{0}$  is a lamination of  $\mathbb{R}^3$  by planes parallel to  $\Pi$ , we conclude that for  $\delta$  and  $\tau$  sufficiently small, the angle of the intersection of any leaf component  $L_V$  of  $\mathcal{L} \cap V$  with any straight line  $\gamma_q$  as above can be made arbitrarily close to  $\frac{\pi}{2}$ . Taking  $\delta$  much smaller than  $\tau$ , a monodromy argument implies that any leaf component  $L_V$  of  $\mathcal{L} \cap V$  which contains a point at distance at most  $\frac{\delta}{2}$  from  $\vec{0}$  is a graph over  $D_{\bar{F}}(\vec{0}, \delta)$ ; in other words,  $\Phi$  restricts to  $L_V$  as a diffeomorphism onto  $D_{\bar{F}}(\vec{0}, \delta)$ .

**Assertion 5.6** *There exists a uniform bound around  $\vec{0}$  for the function  $|\sigma_{\mathcal{F}}|$  defined in (2) (note that this property will complete the proof of Proposition 5.2).*

*Proof.* Reasoning by contradiction, assume that there exists a sequence of points  $p_n$  in leaves  $L_n$  of  $\mathcal{L}$  converging to  $\vec{0}$ , such that  $|\sigma_{L_n}|(p_n)$  diverges. Without loss of generality, we can assume that  $p_n \in V$  and  $p_n$  is a point where the following function attains its maximum:

$$f_n: L_n \cap V \rightarrow [0, \infty), \quad f_n(a) = |\sigma_{L_n}|(a) d_{\bar{F}}(\Phi(a), \partial D_{\bar{F}}(\vec{0}, \delta)),$$

where  $d_{\bar{F}}$  denotes the intrinsic distance in  $\bar{F}$  to the boundary  $\partial D_{\bar{F}}(\vec{0}, \delta)$ . Now expand the above coordinates  $(x, y, z)$  centered at  $\vec{0}$  with ratio  $|\sigma_{L_n}|(p_n) \rightarrow \infty$ . Under this expansion,  $V$  converges to  $\mathbb{R}^3$  with its usual flat metric and the straight lines  $\gamma_q$  converge to parallel lines. The graphical property that  $\Phi$  restricts to any leaf component  $L_V$  of  $\mathcal{L} \cap V$   $\frac{\delta}{2}$ -close to  $\vec{0}$  as a diffeomorphism onto  $D_{\bar{F}}(\vec{0}, \delta)$  gives that after passing to a subsequence, the  $H$ -graphs  $L_n \cap V$  converge after expansion of coordinates to a minimal surface in  $\mathbb{R}^3$  which is an entire graph. By the Bernstein Theorem, such a limit surface is a flat plane. This contradicts that the ratio of the homothetic expansion coincides with the norm of the second fundamental form of  $L_n \cap V$  at  $p_n$  for all  $n$ . This contradiction finishes the proof of Assertion 5.6, and completes the proof of Proposition 5.2.  $\square$

**Proposition 5.7** *Theorem 1.2 holds in the general case for the ambient manifold  $N$ .*

*Proof.* In the manifold setting for  $N$ , under rescaled exponential coordinates from  $p$  we have the same description as in the proof of Proposition 5.2, and the arguments in that proof adapt with straightforward modifications; also see Cases IV and V of the proof of Theorem 1.1 in [16]. This completes the proof of Theorem 1.2.  $\square$

We next extend Corollary 7.1 in [16] to the case of a weak  $H$ -lamination in a Riemannian three-manifold. We remark that the statements in items 4, 5 of the corollary below do not have corresponding statements in Corollary 7.1 in [16]. Regarding item 5 of Corollary 5.9 and using its notation, we make the following definition.

**Definition 5.8** The absolute mean curvature function of a weak CMC foliation  $\mathcal{F}$  of  $N - W$  is the function  $|H_{\mathcal{F}}|: N - W \rightarrow [0, \infty)$  defined by

$$|H_{\mathcal{F}}|(p) = \sup\{|H_L| \mid L \text{ is a leaf of } \mathcal{F} \text{ passing through } p\}.$$

Note that as in the case of  $|\sigma_{\mathcal{F}}|$  given by (2), the function  $|H_{\mathcal{F}}|$  is not necessarily continuous.

The hypothesis of boundedness of  $|H_{\mathcal{F}}|$  in item 5 of Corollary 5.9 is essential: take  $N = \mathbb{R}^3$ ,  $W = \{\vec{0}\}$  and  $\mathcal{F}$  the foliation of  $\mathbb{R}^3 - \{\vec{0}\}$  by concentric spheres.

**Corollary 5.9** *Let  $H \in \mathbb{R}$ . Suppose that  $N$  is a Riemannian three-manifold, not necessarily complete. If  $W \subset N$  is a closed countable subset and  $\mathcal{L}$  is a weak  $H$ -lamination of  $N - W$  such that for every  $p \in W$  there exists positive constants  $\varepsilon, C$  (possibly depending on  $p$ ) satisfying the following curvature estimate:*

$$|\sigma_{\mathcal{L}}|(q) d_N(q, W) \leq C \quad \text{for all } q \in B_N(p, \varepsilon) - W, \quad (4)$$

*then  $\mathcal{L}$  extends across  $W$  to a weak  $H$ -lamination of  $N$ . In particular:*

1. *The closure of any collection of the stable leaves of a weak  $H$ -lamination of  $N - W$  extends across  $W$  to a weak  $H$ -lamination of  $N$  consisting of stable  $H$ -surfaces.*

2. The closure in  $N$  of any collection of limit leaves of a weak  $H$ -lamination  $\mathcal{L}$  of  $N - W$  is a weak  $H$ -lamination of  $N$ , all whose leaves are stable  $H$ -surfaces.
3. If  $\mathcal{F}$  is a weak  $H$ -foliation of  $N - W$ , then  $\mathcal{F}$  extends across  $W$  to a weak  $H$ -foliation of  $N$ .
4. If  $\mathcal{F}$  is a weak CMC foliation of  $N - W$  and  $H \in \mathbb{R}$ , then the closure in  $N - W$  of any collection  $\mathcal{F}(H)$  of leaves of  $\mathcal{F}$  with constant curvature  $H$  extends across  $W$  to a weak  $H$ -lamination of  $N$ .
5. If  $\mathcal{F}$  is a weak CMC foliation of  $N - W$  with bounded absolute mean curvature function, then  $\mathcal{F}$  extends across  $W$  to a weak CMC foliation of  $N$ .

*Proof.* Let  $\mathcal{L}$  be a weak  $H$ -lamination of  $N - W$  satisfying the curvature estimate (4), where  $W$  is closed and countable. Since the extension of  $\mathcal{L}$  across  $W$  is a local question, it suffices to extend  $\mathcal{L}$  in small, open extrinsic balls in  $N$ . Since  $W$  is countable, we can take these balls so that each of their boundaries are disjoint from  $W$ , and their closures in  $N$  are compact. It follows that for every such ball  $B_N$ , the set  $W \cap B_N$  is a complete countable metric space. By Baire's Theorem, the set  $W_0$  of isolated points of the locally compact metric space  $W \cap B_N$  is dense in  $W \cap B_N$ .

**Assertion 5.10** *In the above situation,  $\mathcal{L} \cap B_N$  extends across  $W_0$  to a weak  $H$ -lamination of  $B_N - (W - W_0)$ .*

*Proof.* Consider an isolated point  $p \in W \cap B_N$ . By hypothesis, there exist  $\varepsilon, C > 0$  such that the inequality (4) holds. Taking  $\varepsilon > 0$  smaller if necessary, we can assume that the closed ball  $\overline{B}_N(p, \varepsilon)$  is compact and contained in  $B_N$ , its boundary  $S_N^2(p, \varepsilon)$  is disjoint from  $W$  and that  $B_N(p, \varepsilon) \cap W = \{p\}$ . By Theorem 1.2, the induced local weak  $H$ -lamination  $\mathcal{L} \cap [B_N(p, \varepsilon) - \{p\}]$  extends across  $p$  to a weak  $H$ -lamination of  $B_N(p, \varepsilon)$ . This proves the assertion.  $\square$

Consider the collection  $\mathfrak{U}$  of open subsets  $U$  of  $B_N$  such that  $B_N - W \subset U$  and there exists a weak  $H$ -lamination  $\mathcal{L}_U$  of  $U$  whose restriction to  $B_N - W$  coincides with  $\mathcal{L}|_{B_N - W}$ . By Assertion 5.10,  $B_N - (W - W_0) \in \mathfrak{U}$ . We claim that  $\bigcup_{U \in \mathfrak{U}} U \in \mathfrak{U}$ . Note that if  $U \in \mathfrak{U}$ , then the related weak  $H$ -lamination  $\mathcal{L}_U$  is unique (since leaves of  $\mathcal{L}_U$  are analytic surfaces that coincide with the leaves of  $\mathcal{L}|_{B_N - W}$ ). Given  $U_\alpha, U_\beta \in \mathfrak{U}$  and given a point  $x \in U_\alpha \cap U_\beta$ , the related weak  $H$ -laminations  $\mathcal{L}_\alpha, \mathcal{L}_\beta$  that extend  $\mathcal{L}|_{B_N - W}$  to  $U_\alpha, U_\beta$  satisfy  $\mathcal{L}_\alpha|_{U_\alpha \cap U_\beta} = \mathcal{L}_\beta|_{U_\alpha \cap U_\beta}$  by the above uniqueness property. Therefore,  $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathfrak{U}$  and our claim is proved.

We want to prove that if  $V := \bigcup_{U \in \mathfrak{U}} U \in \mathfrak{U}$ , then  $V = B_N$ , which will finish the proof of the first statement of the corollary. Arguing by contradiction, suppose  $B_N - V \neq \emptyset$ . Since  $B_N - V \subset W \cap B_N$  is a non-empty closed subset of  $W \cap B_N$ , then  $B_N - V$  is a complete countable metric space and so, Baire's theorem again insures that the set  $I$  of its isolated points is dense in  $B_N - V$ . By Assertion 5.10, the  $H$ -lamination  $\mathcal{L}_V$  obtained by extension of  $\mathcal{L}$  to  $V$  extends

through every isolated point of  $B_N - V$ ; hence  $V \cup I \in \mathfrak{U}$ . By definition of  $V$ , this implies that  $V \cup I \subset V$ , hence  $I \subset V$ . As  $I \subset B_N - V$ , then  $I = \emptyset$  which contradicts that  $I$  is dense in  $B_N - V$ . Now the proof of the first statement of Corollary 5.9 is complete.

Item 1 of the corollary follows from the already proven first statement and from curvature estimates for stable  $H$ -surfaces (Schoen [24], Ros [22], see also Theorem 2.15 in [17]). By Theorem 4.3 of [17] (see also Theorem 1 in [18]), limit leaves of a weak  $H$ -lamination are stable (if  $H \neq 0$  they are two-sided; in the minimal case, the two-sided cover of every limit leaf is stable). As the collection of limit leaves of a weak  $H$ -lamination is closed, then item 2 of the corollary follows from item 1. Item 3 is a direct consequence of item 2, as every leaf of a weak  $H$ -foliation is a limit leaf.

To prove item 4, let  $\mathcal{F}$  be a weak CMC foliation of  $N - W$ , where  $W$  is closed and countable, and let  $H \in \mathbb{R}$ . Reasoning as in the case of a weak  $H$ -lamination, we can reduce the proof of the extendability of any collection  $\mathcal{F}(H)$  of leaves of  $\mathcal{F}$  with constant mean curvature  $H$  to the case in which  $\mathcal{F}$  is a weak CMC foliation of a small open extrinsic ball  $B_N$  with compact closure in  $N$ , such that  $W \cap S_N^2 = \emptyset$ . Also the above argument based on Baire's Theorem allows one to reduce the proof to the case that  $B_N = B_N(p, \varepsilon)$  where  $p \in W$  is an isolated point of  $W$  and  $B_N(p, \varepsilon) - \{p\} \subset \text{Int}(N) - W$ . To prove that  $\mathcal{F}(H)$  extends across  $p$ , it suffices to show that for some small  $\varepsilon > 0$ , the induced local weak  $H$ -lamination  $\mathcal{F}_1(H) = \mathcal{F}(H) \cap [B_N(p, \varepsilon) - \{p\}]$  extends across  $p$  to a weak  $H$ -lamination of  $B_N(p, \varepsilon)$ .

Consider the weak CMC foliation  $\mathcal{F}_1 = \mathcal{F} \cap [B_N(p, \varepsilon) - \{p\}]$ . By the universal curvature estimate in Theorem 1.3 applied to each of the compact three-manifolds with boundary  $N(k) = \overline{B_N(p, \varepsilon)} - B_N(p, \frac{\varepsilon}{k})$ ,  $k \in \mathbb{N}$ , there exists a constant  $A > 0$  independent of  $k$  such that for each  $k$ , we have

$$|\sigma_{\mathcal{F}_1}|(q) \leq \frac{A}{\min\{\text{dist}_N(q, \partial N(k)), \frac{\pi}{\sqrt{\Lambda}}\}}, \quad \text{for all } q \in \text{Int}[N(k)], \quad (5)$$

where  $\Lambda \geq 0$  is an upper bound of the sectional curvature of  $N$  in  $\overline{B_N(p, \varepsilon)}$  and  $|\sigma_{\mathcal{F}_1}|(q)$  is defined in (2). Taking  $\varepsilon$  smaller if necessary (this does not change the constant  $\Lambda$ ), we can assume that  $d_N(q, \partial N(k)) \leq \frac{\pi}{\sqrt{\Lambda}}$  for all  $k \in \mathbb{N}$ . Thus, given  $k \geq 3$  and  $q \in N(k) \cap B_N(p, \frac{\varepsilon}{2})$ , we have

$$\begin{aligned} |\sigma_{\mathcal{F}_1}|(q) d_N(q, p) &= |\sigma_{\mathcal{F}_1}|(q) \min\{\text{dist}_N(q, \partial N(k)), \frac{\pi}{\sqrt{\Lambda}}\} \frac{d_N(q, p)}{d_N(q, \partial N(k))} \\ &\stackrel{(5)}{\leq} A \frac{d_N(q, p)}{d_N(q, \partial N(k))} \\ &= A \frac{d_N(q, p)}{d_N(q, S_N^2(p, \varepsilon/k))} \xrightarrow{(k \rightarrow \infty)} A. \end{aligned}$$

Hence, the weak  $H$ -lamination  $\mathcal{F}_1(H)$  satisfies the curvature estimate in the hypothesis of Theorem 1.2, and thus,  $\mathcal{F}_1(H)$  extends across  $p$  as desired. This proves item 4 of the corollary.

Finally we prove item 5. Let  $\mathcal{F}$  be a weak CMC foliation of  $N - W$  with bounded absolute mean curvature function, where  $W$  is closed and countable. Similar arguments as in the previous

cases show that we can reduce the proof of item 5 to the proof of the extendability of a weak CMC foliation  $\mathcal{F}$  of a small open extrinsic ball  $B_N(p, \varepsilon)$  with compact closure in  $N$ , where  $p \in W$  is isolated in  $W$  and  $\overline{B}_N(p, \varepsilon) - \{p\} \subset \text{Int}(N) - W$ . To prove that  $\mathcal{F}$  extends across  $p$ , it suffices to prove that for some smaller  $\varepsilon > 0$ , the induced local weak CMC foliation  $\mathcal{F}_1 = \mathcal{F} \cap [B_N(p, \varepsilon) - \{p\}]$  extends across  $p$  to a weak CMC foliation of  $B_N(p, \varepsilon)$ . Since by hypothesis the absolute mean curvature function of the leaves of  $\mathcal{F}_1$  is bounded, we can choose  $\varepsilon > 0$  sufficiently small so that for all  $\delta \in (0, \varepsilon]$ , the absolute mean curvature function of the (smooth) distance sphere  $S_N^2(p, \delta)$  is strictly greater than the maximum value of the absolute mean curvature of the leaves of  $\mathcal{F}_1$ . As an application of the mean curvature comparison principle, we conclude that the closure of every leaf of  $\mathcal{F}_1$  intersects  $S_N^2(p, \varepsilon)$  (see Claim A in the proof of Lemma 5.1 for a similar argument).

Take a sequence  $\{p_n\}_n \subset B_N(p, \varepsilon) - \{p\}$  converging to  $p$  as  $n \rightarrow \infty$ . As  $\mathcal{F}_1$  is a weak CMC foliation of  $B_N(p, \varepsilon) - \{p\}$ , for each  $n \in \mathbb{N}$  there exists at least one leaf  $L_n$  of  $\mathcal{F}_1$  with  $p_n \in L_n$ . Let  $H_n$  be the (constant) mean curvature of  $L_n$  and let  $H = \limsup H_n$ . After replacing by a subsequence, we may assume that  $H = \lim_n H_n$ . Let  $\mathcal{F}_1(H)$  be the weak  $H$ -lamination of  $B_N(p, \varepsilon) - \{p\}$  consisting of all leaves of  $\mathcal{F}_1$  whose mean curvature is  $H$ .

We claim that  $p$  lies in the closure of  $\mathcal{F}_1(H)$  in  $B_N(p, \varepsilon)$ . To see this it suffices to show that given  $k \in \mathbb{N}$ , some leaf of  $\mathcal{F}_1(H)$  intersects  $S_N^2(p, \frac{\varepsilon}{k})$ . Fix  $k \in \mathbb{N}$ . As  $\{p_n\}_n \rightarrow p$ , then for  $n$  sufficiently large  $p_n \in B_N(p, \frac{\varepsilon}{k})$ . As the closure of  $L_n$  intersects  $S_N^2(p, \varepsilon)$  and  $L_n$  is connected, then  $L_n$  also intersects  $S_N^2(p, \frac{\varepsilon}{k})$ . For each  $n \in \mathbb{N}$  large, pick a point  $x_n \in L_n \cap S_N^2(p, \frac{\varepsilon}{k})$ . Since  $S_N^2(p, \frac{\varepsilon}{k})$  is compact, after extracting a subsequence, the  $x_n$  converge as  $n \rightarrow \infty$  to a point  $x \in S_N^2(p, \frac{\varepsilon}{k})$ . As the mean curvatures of the  $L_n$  converge to  $H$ , then there passes a leaf  $\hat{L}$  of  $\mathcal{F}_1(H)$  through  $x$ , and our claim is proved.

**Assertion 5.11** *The weak CMC foliation  $\mathcal{F}_1$  extends across  $p$  to a weak CMC foliation of  $B_N(p, \varepsilon)$  (and thus, the proof of item 5 of Corollary 5.9 is complete).*

*Proof.* By the last claim and the already proven item 4 of this corollary,  $\mathcal{F}_1(H)$  extends across  $p$  to a weak  $H$ -lamination of  $B_N(p, \varepsilon)$ . Let  $\bar{L}$  be the leaf of the extended weak  $H$ -lamination  $\mathcal{F}_1(H) \cup \{p\}$  passing through  $p$  (thus,  $L = \bar{L} - \{p\}$  is a leaf of  $\mathcal{F}_1$ ). After possibly choosing a smaller  $\varepsilon$ , we may assume that  $\bar{L}$  is a smooth embedded disk in  $\overline{B}_N(p, \varepsilon)$  with compact boundary in  $S_N^2(p, \varepsilon)$ . Using again the curvature estimates (5) we get that for any sequence of positive numbers  $\lambda_n \rightarrow \infty$ , a subsequence of the punctured balls  $\lambda_n[B_N(p, \varepsilon) - \{p\}]$  converges as  $n \rightarrow \infty$  to  $\mathbb{R}^3 - \{\vec{0}\}$  with its usual metric, and the weak CMC foliations  $\lambda_n \mathcal{F}_1$  converge to a limit weak CMC foliation  $\mathcal{F}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , which is in fact a minimal foliation since the mean curvature of the leaves of  $\mathcal{F}_1$  is bounded. Note that one of the leaves of  $\mathcal{F}_\infty$  is the punctured plane  $\Pi$  passing through  $\vec{0}$ , corresponding to the blow-up of the tangent plane to the disk  $\bar{L}$  at  $p$ . By item 3 of this corollary,  $\mathcal{F}_\infty$  extends across the origin to a minimal foliation of  $\mathbb{R}^3$ ; since every leaf of this extended minimal foliation is a complete stable minimal surface in  $\mathbb{R}^3$ , then every such leaf is a plane, which must be parallel to  $\Pi$ . In particular, the limit foliation  $\mathcal{F}_\infty$  is independent of

the sequence  $\lambda_n \rightarrow \infty$ . In this situation, it follows that for  $\varepsilon$  sufficiently small, the leaves in  $\mathcal{F}_1$  can be uniformly locally expressed as non-negative or non-positive normal graphs with bounded gradient over their projections to  $\bar{L}$ . In particular, there is a weak CMC foliation structure on  $\mathcal{F}_1 \cup \{p\}$ , and Assertion 5.11 is proved.  $\square$

## 6 Proof of Theorem 1.1.

We start by proving the  $\mathbb{R}^3$ -version of Theorem 1.1 in the more general setting of weak CMC foliations.

**Theorem 6.1** *Suppose that  $\mathcal{F}$  is a weak CMC foliation of  $\mathbb{R}^3$  with a closed countable set  $\mathcal{S}$  of singularities (these are the points where the weak CMC structure of  $\mathcal{F}$  cannot be extended). Then, each leaf of  $\mathcal{F}$  is contained in either a plane or a round sphere, and  $0 \leq |\mathcal{S}| \leq 2$ . Furthermore if  $\mathcal{S}$  is empty, then  $\mathcal{F}$  is a foliation by planes.*

*Proof.* Note that if all leaves of  $\mathcal{F}$  are minimal, then  $\mathcal{F}$  is a minimal foliation of  $\mathbb{R}^3 - \mathcal{S}$ , hence by item 3 of Corollary 5.9,  $\mathcal{F}$  extends to a minimal foliation of  $\mathbb{R}^3$ , which must then consist entirely of parallel planes and the theorem holds in this case. Therefore, in the sequel we may assume that  $\mathcal{F}$  contains a leaf which is not minimal.

**Assertion 6.2** *Every non-minimal leaf of  $\mathcal{F}$  is proper in  $\mathbb{R}^3 - \mathcal{S}$ . Furthermore, if  $\mathcal{S}$  is bounded, then every non-minimal leaf of  $\mathcal{F}$  is contained in a ball.*

*Proof.* Consider a leaf  $L$  of  $\mathcal{F}$ , with mean curvature  $H$ . By item 4 of Corollary 5.9, the collection  $\mathcal{F}(H)$  of  $H$ -leaves in  $\mathcal{F}$  extends across  $\mathcal{S}$  to weak  $H$ -lamination of  $\mathbb{R}^3$  and so, the closure of  $L$  in  $\mathbb{R}^3$  is a weak  $H$ -lamination of  $\mathbb{R}^3$ . If  $H \neq 0$  and  $L$  is not proper in  $\mathbb{R}^3 - \mathcal{S}$ , then  $\bar{L}$  contains a limit leaf  $L_1$ , which is complete since  $\bar{L}$  is a weak  $H$ -lamination of  $\mathbb{R}^3$ . By Theorem 4.3 of [17] (see also Theorem 1 in [18]) applied to the weak  $H$ -lamination  $\bar{L}$ ,  $L_1$  is stable ( $L_1$  is two-sided since its mean curvature is non-zero). This contradicts that there are no stable complete  $H$ -surfaces in  $\mathbb{R}^3$  for any  $H \neq 0$ . This proves the first sentence in the assertion.

Next suppose  $\mathcal{S}$  is bounded and take a non-minimal leaf  $L \in \mathcal{F}$ . If there exists an extrinsically divergent sequence of points  $p_n \in L$ , then the extrinsic distance  $d_n$  from  $p_n$  to  $\mathcal{S}$  tends to infinity as  $n \rightarrow \infty$  and hence, Corollary 4.1 applied to the weak CMC foliation  $\mathcal{F} \cap \mathbb{B}(p_n, d_n)$  of  $\mathbb{B}(p_n, d_n)$  implies that the second fundamental form of  $L$  at  $p_n$  decays to zero in norm, as  $|\sigma_L| \leq |\sigma_{\mathcal{F}}|$ . In particular, the trace of the second fundamental form of  $L$  must be zero since  $L$  has constant mean curvature, which gives a contradiction. Therefore, every non-minimal leaf  $L$  of  $\mathcal{F}$  lies in some ball of  $\mathbb{R}^3$ .  $\square$

**Assertion 6.3** *If a leaf  $L$  of  $\mathcal{F}$  is contained in a ball of  $\mathbb{R}^3$ , then its closure  $\bar{L}$  is a round sphere.*

*Proof.* As in the proof of the previous assertion, the closure of  $L$  in  $\mathbb{R}^3$  has the structure of a weak  $H$ -lamination of  $\mathbb{R}^3$  by item 4 of Corollary 5.9. Since there are no bounded minimal laminations in  $\mathbb{R}^3$  by the maximum principle, then  $H \neq 0$ . By Assertion 6.2,  $L$  is proper in  $\mathbb{R}^3 - \mathcal{S}$ , and thus,  $\bar{L}$  consists of a single compact immersed surface which does not intersect itself transversely, and whenever  $\bar{L}$  intersects itself, it locally consists of two disks with opposite mean curvature vectors. Hence,  $\bar{L}$  is Alexandrov-embedded. In this situation, Alexandrov [1] proved that  $\bar{L}$  is a round sphere.  $\square$

**Assertion 6.4** *If  $\mathcal{S}$  is bounded, then Theorem 6.1 hold.*

*Proof.* As  $\mathcal{S}$  is bounded, then Assertions 6.2 and 6.3 give that the closure of every non-minimal leaf of  $\mathcal{F}$  is a round sphere. Consider the collection  $\mathcal{A}$  of all spherical leaves of  $\mathcal{F}$  union with  $\mathcal{S}$ . Then, the restriction of  $\mathcal{F}$  to the complement of the closure of  $\mathcal{A}$  is a minimal foliation  $\mathcal{F}_1$  of the open set  $\mathbb{R}^3 - (\bar{\mathcal{A}} \cup \mathcal{S})$ . Applying item 3 of Corollary 5.9 to  $\mathcal{F}_1$ ,  $N = \mathbb{R}^3 - \bar{\mathcal{A}}$  and  $W = \mathcal{S} \cap N$ , we conclude that  $\mathcal{F}_1$  extends across  $\mathcal{S} \cap N$  to a minimal foliation of  $\mathbb{R}^3 - \bar{\mathcal{A}}$ . By item 4 of Corollary 5.9,  $\mathcal{F}_1$  extend across  $\mathcal{S}$  to a minimal lamination of  $\mathbb{R}^3$ , and thus the extended leaves of  $\mathcal{F}_1$  are complete. As  $\mathcal{F}_1$  consists of stable leaves by Theorem 1 in [18], then item 1 of Corollary 5.9 insures that the extended leaves of  $\mathcal{F}$  across  $\mathcal{S}$  are complete stable minimal surfaces in  $\mathbb{R}^3$ , hence planes. As the weak CMC foliation  $\mathcal{F}$  is now entirely formed by punctured spheres and planes, then it is clear that  $\mathcal{S}$  consists of one of two points. This completes the proof of Assertion 6.4.  $\square$

To prove Theorem 6.1 in the general case of a closed countable set  $\mathcal{S} \subset \mathbb{R}^3$ , we next analyze the structure of  $\mathcal{F}$  in a neighborhood of an isolated point  $p \in \mathcal{S}$  (recall that the set of isolated points in  $\mathcal{S}$  is dense in  $\mathcal{S}$  by Baire's Theorem). Since  $p$  is isolated in  $\mathcal{S}$ , we can choose a sphere  $\mathbb{S}^2(p, r)$  such that  $\bar{\mathbb{B}}(p, r) \cap \mathcal{S} = \{p\}$ . As  $\mathcal{S}$  is closed, then  $\mathbb{S}^2(p, r)$  is at positive distance from  $\mathcal{S}$ . Since  $|\sigma_{\mathcal{F}}|$  is locally bounded in  $\mathbb{R}^3 - \mathcal{S}$  (by definition of weak CMC lamination), an elementary compactness argument shows that there is a uniform upper bound for the restriction to  $\mathbb{S}^2(p, r)$  of the norms of the second fundamental forms of all leaves in  $\mathcal{F}$  which intersect  $\mathbb{S}^2(p, r)$ ; in particular the absolute mean curvature of every such leaf satisfies  $|H| \leq C_1$  for some  $C_1 > 0$ . By item 5 of Corollary 5.9, the mean curvature of the leaves of  $\mathcal{F}$  is unbounded in every neighborhood of  $p$ , since  $p \in \mathcal{S}$ . Therefore, there exist leaves of  $\mathcal{F}$  which intersect  $\mathbb{B}(p, r)$  and whose mean curvatures satisfy  $|H| > C_1$ . Every such leaf  $L$  is entirely contained in  $\mathbb{B}(p, r)$  and thus, Assertion 6.3 implies that the closure  $\bar{L}$  of  $L$  in  $\mathbb{R}^3$  is a round sphere. Note that either  $p \in \bar{L}$  or  $p$  lies in the open ball  $\mathbb{B}_L$  enclosed by  $\bar{L}$  (otherwise a monodromy argument shows that  $\mathcal{F} \cap \mathbb{B}_L$  is a “product” foliation by spheres, which produces a singularity  $q \in \mathbb{B}_L$ ; this contradicts that  $\mathcal{S} \cap \mathbb{B}(p, r) = \{p\}$ ).

The above arguments show that for every isolated point  $p$  of  $\mathcal{S}$ , one of the two following possibilities holds:



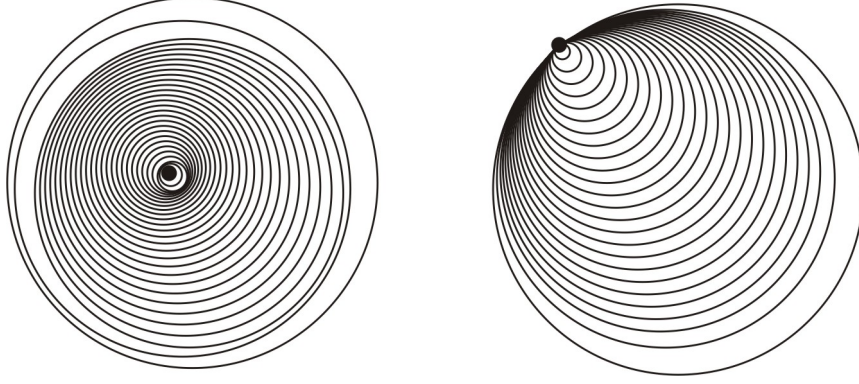


Figure 4: Left: Case **(A)** of the proof of Theorem 6.1. Right: Case **(B)**. In both cases, the dot represents an isolated singular point  $p \in \mathcal{S}$ .

- (A)** There exists an open neighborhood  $V_p$  of  $p$  in  $\mathbb{R}^3$  such that  $\mathcal{F}$  restricts to  $V_p - \{p\}$  as a weak CMC foliation by round spheres and  $V_p \cap \mathcal{S} = \{p\}$ , see Figure 4 left.
- (B)** There exists an open ball  $\mathbb{B}(q, R) \subset \mathbb{R}^3$  such that  $p \in \mathbb{S}^2(q, R)$  and the weak CMC foliation  $\mathcal{F}$  restricts to  $\overline{\mathbb{B}}(q, R) - \{p\}$  as a union of round spheres punctured at  $p$ , all tangent at  $p$ . In this case, we call  $V_p = \mathbb{B}(q, R)$ , see Figure 4 right.

If possibility **(A)** occurs for an isolated point  $p$  of  $\mathcal{S}$ , we define  $U_p$  to be the maximal such open set  $V_p$ , with the ordering given by the inclusion. Note that in this case, we have two mutually exclusive possibilities:

- (A1)** The boundary  $\partial U_p$  of  $U_p$  is empty; in this case,  $U_p = \mathbb{R}^3$  and Theorem 6.1 is proved with  $\mathcal{S} = \{p\}$ .
- (A2)** The boundary  $\partial U_p$  is non-empty; in this case  $\partial U_p$  is either a round sphere (and  $U_p$  is an open ball of  $\mathbb{R}^3$  containing  $p$ ), or  $\partial U_p$  is a plane (and  $U_p$  is an open half-space containing  $p$ ). In both of these cases,  $U_p$  only intersects  $\mathcal{S}$  at the point  $p$ .

If possibility **(B)** holds for an isolated point  $p$  of  $\mathcal{S}$ , we define  $U_p$  to be the union of the maximal open 1-parameter family of spheres in  $\mathcal{F}$ , possibly punctured at  $p$ , that contains the open ball  $V_p$  described in possibility **(B)**, together with the point  $p$  if this union contains a spherical leaf of  $\mathcal{F}$  that does not pass through  $p$ . As in case **(A)**, we have two mutually exclusive possibilities:

- (B1)** The boundary  $\partial U_p$  of  $U_p$  is empty; in this case,  $U_p = \mathbb{R}^3$  and Theorem 6.1 is again proved with  $\mathcal{S} = \{p\}$ . Therefore, in the sequel we will assume that for each isolated point  $p \in \mathcal{S}$ , we have  $\partial U_p \neq \emptyset$ .

**(B2)** The boundary  $\partial U_p$  is non-empty; in this case  $\partial U_p$  is either a round sphere (and in  $U_p$  is an open ball of  $\mathbb{R}^3$  with  $p$  in its closure), or  $\partial U_p$  is a plane (and  $U_p$  is an open halfspace with  $p$  in its closure). In both of these cases,  $U_p$  only intersects  $\mathcal{S}$  in at most the point  $p$ .

Note that the case of two simultaneous such maximal open sets  $U_p \neq U'_p$  can occur in case **(B)**; for instance when  $\mathcal{F}$  is the foliation of  $\mathbb{R}^3 - \{\vec{0}\}$  given by the  $(x_1, x_2)$ -plane together with all spheres passing through  $p = \vec{0}$  and tangent to the  $(x_1, x_2)$ -plane (in this case  $U_p = \{x_3 > 0\}$  and  $U'_p = \{x_3 < 0\}$ ). In the case that we have two possibilities for choosing  $U_p$ , we will simply arbitrarily choose one such  $U_p$  in our discussions below.

We next collect some elementary properties of these open sets  $U_p$ , which easily follow from the fact that  $\mathcal{F}$  is a foliation outside  $\mathcal{S}$  and from the description in possibilities **(A)**, **(B)** above.

**(P1)** If  $p, q$  are distinct isolated points of  $\mathcal{S}$ , then  $U_p \cap U_q = \emptyset$ .

**(P2)** If  $\{p_n\}_n$  is a converging sequence of distinct isolated points of  $\mathcal{S}$ , then for  $n$  sufficiently large,  $U_{p_n}$  is an open ball and the radii of the  $U_{p_n}$  converge to zero.

Note that by maximality and a standard monodromy argument, if  $\partial U_p$  is a sphere then  $\partial U_p \cap \mathcal{S} \neq \emptyset$ .

**Assertion 6.5** *Given an isolated point  $p \in \mathcal{S}$ , suppose that  $U_p$  is an open ball. Then,  $\partial U_p \cap \mathcal{S}$  contains at least one point which is not isolated in  $\mathcal{S}$ .*

*Proof.* Recall that  $\partial U_p \cap \mathcal{S} \neq \emptyset$ . Arguing by contradiction, suppose that  $\partial U_p \cap \mathcal{S}$  consists only of isolated points of  $\mathcal{S}$ . In particular,  $\partial U_p \cap \mathcal{S}$  is finite, say  $\partial U_p \cap \mathcal{S} = \{p_1, \dots, p_k\}$ . Note that  $p$  lies in  $\partial U_p \cap \mathcal{S}$  if and only if possibility **(B)** above holds for  $p$ . Then, the above arguments show that around every point  $p_1 \in \partial U_p \cap \mathcal{S}$ , necessarily Case **(B)** occurs (exchanging  $p$  by  $p_1$ ), and that the following additional property holds:

**(P3)** If  $p_1 \neq p$ , then the related maximal open set  $U_{p_1}$  is disjoint from  $U_p$  and every leaf in the restriction of  $\mathcal{F}$  to  $\overline{U_{p_1}}$  is a punctured sphere or punctured plane whose closure only intersects  $\overline{U_p}$  at  $p_1$ .

Analogously, if  $p \in \partial U_p \cap \mathcal{S}$  is an isolated point where we have two possibilities  $U_p, U'_p$  for choosing  $U_p$ , then the same property **(P3)** holds for  $p_1 = p$  and  $U_{p_1} = U'_p$ , see Figure 5.

Let  $B_1, \dots, B_k \subset \mathbb{R}^3$  be pairwise disjoint, small open balls centered at the points  $p_1, \dots, p_k$ . As the  $p_i$  are isolated in  $\mathcal{S}$ , we can assume that  $\mathcal{S} \cap \left(\bigcup_{i=1}^k B_i\right) = \partial U_p \cap \mathcal{S}$ . We denote by  $D_i = B_i \cap \partial U_p$ , which is a spherical disk. Next we will prove the following property.

**(P4)** Given  $i = 1, \dots, k$ , if the radius of the ball  $B_i$  is small enough, then the intersection of  $\mathcal{F}$  with the region  $W_i = B_i - [U_{p_i} \cup U_p]$  consists of a collection of annuli, each of which can be expressed as a normal graph over its projection to  $D_i - \{p_i\}$ .

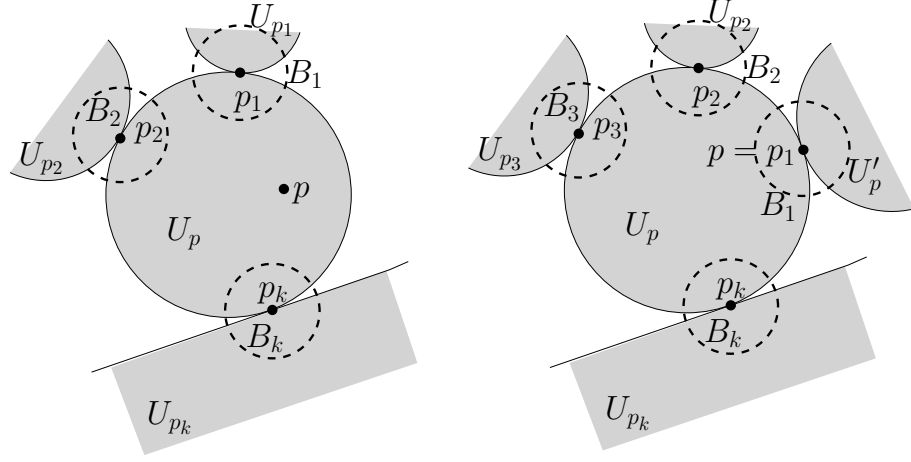


Figure 5: On the left, possibility **(A)** occurs for  $p$ . On the right, possibility **(B)** holds for  $p$  and there are two possible choices  $U_p, U'_p$  for  $U_p$ . In both figures, we have represented one of the points in  $\partial U_p \cap \mathcal{S}$  (namely,  $p_k$ ) so that the related set  $U_{p_k}$  is a halfspace.

To see why **(P4)** holds, we first prove that under blow-up from  $p_i$ , the induced foliation  $\mathcal{F} \cap W_i$  converges smoothly to the punctured tangent plane to  $\partial U_p$  at  $p_i$ . Note that the region  $W_i$  converges after such a blow-up to the punctured tangent plane to  $\partial U_p$  at  $p_i$ . Since we have a scale-invariant uniform bound on the second fundamental of  $(\mathcal{F} \cap B_i) - \{p_i\}$  (given by Theorem 1.3, see the proof of Assertion 6.2 for a similar argument), then the leaves of the induced foliation  $\mathcal{F} \cap W_i$  are locally graphical over small geodesic disks of  $\partial U_p - \{p_i\}$  contained in  $D_i$ . It remains to check that these local graphs, when glued together inside a leaf of  $\mathcal{F} \cap W_i$ , do not define multi-valued graphs over the punctured spherical disk  $D_i - \{p_i\}$ . Arguing by contradiction, suppose that there exists a leaf  $L_{W_i}$  of  $\mathcal{F} \cap W_i$  which can be expressed by a multi-valued graph (not univalent) over  $D_i - \{p_i\}$ . Consider a small compact solid cylinder  $C_i$  whose axis is the normal line to  $\partial U_p$  passing through  $p_i$ , such that both  $\partial U_p$  and  $\partial U_{p_i}$  intersect  $C_i$  in compact closed disks bounded by distinct parallel circles. Then, the intersection of  $L_{W_i}$  with  $\partial C$  contains a spiraling curve  $\Gamma$  (with infinite length in both directions) which is trapped between the circles  $\partial U_p \cap C$ ,  $\partial U_{p_i} \cap C$ . The curve  $\Gamma$  limits to two disjoint closed curves  $\widehat{\Gamma}_1, \widehat{\Gamma}_2 \subset \partial C$ , which are topologically parallel to  $\partial U_p \cap C$ ,  $\partial U_{p_i} \cap C$ . Let  $\widehat{L}_1, \widehat{L}_2$  be the leaves of  $\mathcal{F}$  that contain  $\widehat{\Gamma}_1, \widehat{\Gamma}_2$ , respectively (note that a priori,  $\widehat{L}_1$  could coincide with  $\widehat{L}_2$ ). As  $\widehat{\Gamma}_1, \widehat{\Gamma}_2$  consist of limit points of  $\Gamma$ , then  $\widehat{L}_1, \widehat{L}_2$  are limit leaves of the sublamination  $\mathcal{F}(H)$  of  $\mathcal{F}$  consisting of the leaves of  $\mathcal{F}$  with the same mean curvature  $H$  as  $L_{W_i}$ . By item 4 of Corollary 5.9, both  $\widehat{L}_1, \widehat{L}_2$  extend smoothly across  $p_i$ . Since the  $H$ -surfaces  $\widehat{L}_1 \cup \{p_i\}, \widehat{L}_2 \cup \{p_i\}$  have the same mean curvature vectors at  $p_i$  (their orientations are induced by the one of the multi-valued graph inside  $L_{W_i}$ ), we contradict

the maximum principle for  $H$ -surfaces. Now property **(P4)** is proved.

Since **(P4)** holds, it follows that  $\partial U_p - \mathcal{S}$ , considered to be a leaf of the weak CMC foliation  $\mathcal{F}$  of  $\mathbb{R}^3 - \mathcal{S}$ , has trivial holonomy on the exterior side of  $\partial U_p$ . This implies that the leaves of  $\mathcal{F}$  nearby  $\partial U_p - \mathcal{S}$  and outside  $U_p$  are topologically punctured spheres which are graphs over  $\partial U_p - \mathcal{S}$ . Therefore, these graphs extend smoothly to embedded topological spheres with constant mean curvature; hence the extended graphs are themselves round spheres. This contradicts the maximality of  $U_p$ . This contradiction finishes the proof of Assertion 6.5.  $\square$

**Assertion 6.6** *Theorem 6.1 holds.*

*Proof.* Consider the set  $\mathcal{S}_0$  of those points of  $\mathcal{S}$  which are isolated; recall that  $\mathcal{S}_0$  is an open dense subset of  $\mathcal{S}$  by Baire's Theorem. If for every  $p \in \mathcal{S}_0$  the related maximal open set  $U_p$  given just before Assertion 6.5 is a halfspace, then clearly  $\mathcal{S}$  only consists of one or two singularities, and by Assertion 6.4, the theorem holds in this case. So it suffices to show that for every  $p \in \mathcal{S}_0$ ,  $U_p$  cannot be an open ball. Arguing by contradiction, suppose that there exists  $p \in \mathcal{S}_0$  such that  $U_p$  is an open ball. Since  $\mathcal{S} - \mathcal{S}_0 \neq \emptyset$  by Assertion 6.5 and  $\mathcal{S} - \mathcal{S}_0$  is closed in  $\mathcal{S}$ , then the set  $\mathcal{S}_1$  of isolated points in  $\mathcal{S} - \mathcal{S}_0$  is non-empty (in fact,  $\mathcal{S}_1$  is dense in  $\mathcal{S} - \mathcal{S}_0$  by Baire's Theorem applied to the complete metric space  $\mathcal{S} - \mathcal{S}_0$  together with the collection of open dense subsets of  $\mathcal{S} - \mathcal{S}_0$  given by  $A_n = (\mathcal{S} - \mathcal{S}_0) - \{p_1, \dots, p_n\}$ , where  $\{p_n \mid n \in \mathbb{N}\}$  is an enumeration of the countable set  $\mathcal{S} - (\mathcal{S}_0 \cup \mathcal{S}_1)$ ). Pick a point  $q \in \mathcal{S}_1$ , which must be a limit of a sequence of points  $p_n \in \mathcal{S}_0$ . Since the  $p_n$  converge to  $q$ , property **(P2)** insures that for  $n$  large,  $U_{p_n}$  is an open ball and the radii of  $U_{p_n}$  converge to zero as  $n \rightarrow \infty$ . Since each  $\partial U_{p_n} \cap \mathcal{S}$  contains at least one point in  $\mathcal{S} - \mathcal{S}_0$  (by Assertion 6.5), then we contradict that  $q$  is isolated in  $\mathcal{S} - \mathcal{S}_0$ . This contradiction proves that  $U_p$  cannot be an open ball, and therefore finishes the proof of Assertion 6.6.  $\square$

We next prove the  $\mathbb{S}^3$ -version of Theorem 6.1.

**Theorem 6.7** *Let  $\mathcal{F}$  be a weak CMC foliation of  $\mathbb{S}^3$  with a closed countable set  $\mathcal{S}$  of singularities (as in Theorem 6.1, these singularities are the points where the weak CMC foliation structure of  $\mathcal{F}$  cannot be extended). Then, each leaf of  $\mathcal{F}$  is contained in a round sphere in  $\mathbb{S}^3$  and the number of singularities is  $|\mathcal{S}| = 1$  or  $2$ .*

*Proof.* This theorem can be proven with minor modifications of the proof in the  $\mathbb{R}^3$  case. One starts by proving that if  $\mathcal{F}$  is a weak CMC foliation of  $\mathbb{S}^3 - \mathcal{S}$  with  $\mathcal{S}$  closed, then  $\mathcal{S}$  cannot be empty. This holds since otherwise, as  $\mathbb{S}^3$  is compact and the absolute mean curvature function of  $\mathcal{F}$  is locally bounded, there exists a leaf in  $\mathcal{F}$  of maximal absolute mean curvature, which by Proposition 5.4 in [17] must be stable. This contradicts the non-existence of complete, stable surfaces with constant mean curvature in  $\mathbb{S}^3$ .

By item 4 of Corollary 5.9, for any  $H \in \mathbb{R}$ , the weak  $H$ -lamination  $\mathcal{F}(H)$  of  $\mathcal{F}$  consisting of all leaves of constant mean curvature  $H$  extends smoothly across the singular set  $\mathcal{S}$  of  $\mathcal{F}$  to

a collection  $\mathcal{F}(H)' = \{\bar{L} \mid L \in \mathcal{F}(H)\}$  of compact immersed  $H$ -surfaces. The non-existence of limit leaves of  $\mathcal{F}(H)'$  implies that this collection of surfaces is finite.

Next we show that  $\bar{L}$  is Alexandrov-embedded for every leaf  $L$  of  $\mathcal{F}(H)$ . Observe that  $\bar{L}$  cannot have transversal intersections, as  $\mathcal{F}$  is a weak CMC foliation. If  $\bar{L}$  is not embedded, then  $H \neq 0$  and there exists  $p \in \bar{L}$  such that locally around  $p$ ,  $\bar{L}$  consists of two disks that lie at one side of each other with opposite mean curvature vectors. As  $\bar{L}$  is compact, there exists some small  $\varepsilon > 0$  such that

$$\bar{L}(\varepsilon) = \{\exp_p(tN_p) \mid p \in \bar{L}, t \in [0, \varepsilon)\}$$

is an embedded  $\varepsilon$ -neighborhood on the mean convex side of  $\bar{L}$ , where  $N$  stands for the unit normal vector field to  $\bar{L}$  for which  $H$  is the mean curvature of  $\bar{L}$ . In particular, the parallel surface  $\bar{L}_{\varepsilon/2}$  at distance  $\frac{\varepsilon}{2}$  from  $\bar{L}$  inside  $\bar{L}(\varepsilon)$  is embedded. As every compact embedded surface separates  $\mathbb{S}^3$ , then  $\bar{L}_{\varepsilon/2}$  divides  $\mathbb{S}^3$  into two open domains, one of which, called  $\Omega$ , contains  $\partial\bar{L}(\varepsilon) - \bar{L}$ . The union of the closure of  $\Omega$  in  $\mathbb{S}^3$  with the  $\frac{\varepsilon}{2}$ -neighborhood  $\bar{L}(\frac{\varepsilon}{2}) \subset \bar{L}(\varepsilon)$  of  $\bar{L}$  can be viewed as the image of a submersion of a 3-manifold with boundary into  $\mathbb{S}^3$ , with its boundary image being  $\bar{L}$ . This proves that  $\bar{L}$  is Alexandrov-embedded.

Let  $p \in \mathcal{S}$  be an isolated point in  $\mathcal{S}$ , which exists by Baire's Theorem. As  $p$  is isolated in  $\mathcal{S}$ , we can choose a geodesic sphere  $S^2(p, r)$  in  $\mathbb{S}^3$  of small radius  $r \in (0, \frac{\pi}{2})$  centered at  $p$  such that the geodesic ball  $B(p, r)$  enclosed by  $S^2(p, r)$  satisfies  $B(p, r) \cap \mathcal{S} = \{p\}$ . As  $\mathcal{S}$  is closed, then  $S^2(p, r)$  is at positive distance from  $\mathcal{S}$ . Since  $|\sigma_{\mathcal{F}}|$  is locally bounded in  $\mathbb{S}^3 - \mathcal{S}$ , an elementary compactness argument shows that there is a uniform upper bound for the restriction to  $S^2(p, r)$  of the norms of the second fundamental forms of all leaves in  $\mathcal{F}$  which intersect  $S^2(p, r)$ ; in particular the absolute mean curvature of every such leaf satisfies  $|H| \leq C$  for some  $C > 0$ . By item 5 of Corollary 5.9, the mean curvature of the leaves of  $\mathcal{F}$  is unbounded in every neighborhood of  $p$ , since  $p \in \mathcal{S}$ . Therefore, there exist leaves of  $\mathcal{F}$  which intersect  $B(p, r)$  and whose mean curvatures satisfy  $|H| > C$ . Every such leaf  $L$  is then entirely contained in  $B(p, r)$ . As  $r < \frac{\pi}{2}$ , then  $\bar{L}$  is contained in a hemisphere, and by a standard application of the Alexandrov moving plane technique, we conclude that  $\bar{L}$  is a sphere. As we did in the case of  $\mathbb{R}^3$ , one can consider the maximal collection  $U_p$  of round spheres in  $\mathcal{F}$  around every isolated point of  $\mathcal{S}$ . In this setting, Assertion 6.5 remains valid with only straightforward modifications in its proof, as well as the Baire's Theorem argument in the proof of Assertion 6.6. We leave the remaining details for the reader.  $\square$

**Remark 6.8** Recall that by Theorem 6.1, if a weak CMC foliation  $\mathcal{F}$  of  $\mathbb{R}^3$  has a countable set of singularities  $\mathcal{S}$  (as a weak foliation), then  $\mathcal{S}$  consists of at most two points. If  $|\mathcal{S}| = 0$ , then  $\mathcal{F}$  is a foliations by parallel planes. If  $|\mathcal{S}| = 1$ , then up to a translation we can assume  $\mathcal{S} = \{\vec{0}\}$  and we have two cases:

1. Some leaf of  $\mathcal{F}$  has the origin in its closure. In this case, up to a rotation of  $\mathbb{R}^3$  fixing  $\vec{0}$ , there are exactly two examples  $\mathcal{F}_1, \mathcal{F}_2$ , where  $\mathcal{F}_1$  is the set of horizontal planes  $\{x_3 = t \mid t \leq 0\}$

together with the spheres  $\mathbb{S}^2(p_t, t)$  with  $p_t = (0, 0, t)$  for all  $t > 0$ , and  $\mathcal{F}_2$  is the set of spheres tangent to the  $(x_1, x_2)$ -plane at  $\vec{0}$  together with the  $(x_1, x_2)$ -plane. In particular, in this Case 1 the weak foliation  $\mathcal{F}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  is actually a foliation of  $\mathbb{R}^3 - \{\vec{0}\}$ .

2. No leaf of  $\mathcal{F}$  has the origin in its closure, as in the particular foliation  $\mathcal{F}_0$  of  $\mathbb{R}^3 - \{\vec{0}\}$  consisting of the set of spheres centered at the origin. In this Case 2, the weak foliation  $\mathcal{F}$  can be isotoped to  $\mathcal{F}_0$  fixing the origin along the isotopy, and the spherical leaves can intersect themselves as leaves of a weak foliation.

In Theorem 7.1 below we will describe the structure of a singular weak CMC foliation  $\mathcal{F}$  of a three-manifold in a neighborhood of an isolated singularity  $p$  as being modeled on a weak singular CMC foliation of  $\mathbb{R}^3 - \{\vec{0}\}$  with  $|\mathcal{S}| = 1$ ; in fact we will show that the weak foliation  $\mathcal{F}$  is closely approximated by exactly one of the examples in the two cases above.

## 7 Structure of singular CMC foliations in a neighborhood of an isolated singularity.

Theorem 6.1 implies that the two possibilities represented in Figure 4 give canonical models for every weak CMC foliation of  $\mathbb{R}^3 - \{\vec{0}\}$  as a 1-parameter collection of spheres and planes with one singularity occurring at the origin. In fact, this description is a good model for the local structure of any weak CMC foliation in a Riemannian three-manifold around an isolated singularity, as the following theorem demonstrates.

**Theorem 7.1** *Let  $B_N(p, r)$  be a metric ball in a Riemannian three-manifold and  $\mathcal{F}$  be a weak CMC foliation of  $B_N(p, r) - \{p\}$ . Then:*

1.  $\mathcal{F}$  extends across  $p$  to a weak CMC foliation of  $B_N(p, r)$  if and only if the absolute mean curvature function of  $\mathcal{F}$  (see Definition 5.8) is bounded.
2. Suppose that the absolute mean curvature function of  $\mathcal{F}$  is unbounded in  $B_N(p, \frac{r}{2})$ . Then, there exists  $r_0 \in (0, r)$  such that:
  - (2A) For every sequence  $l_n > 0$  with  $l_n \rightarrow \infty$ , the blow-up weak CMC foliations  $l_n[\mathcal{F} \cap B_N(p, r_0)]$  of  $l_n B_N(p, r_0) - \{p\}$  converge (after extracting a subsequence) to a non-flat weak CMC foliation of  $\mathbb{R}^3 - \{\vec{0}\}$ .
  - (2B) There exists  $H_0 > 0$  such that the closure in  $B_N(p, r_0)$  of every leaf of  $\mathcal{F} \cap B_N(p, r_0)$  with absolute mean curvature  $H > H_0$  is an embedded  $H$ -sphere that bounds a subball of  $B_N(p, r)$  that contains  $p$  in its closure.

*Proof.* Item 1 of this theorem follows directly from item 5 of Corollary 5.9.

In the sequel we will assume that the absolute mean curvature function of  $\mathcal{F}$  is unbounded in  $B_N(p, \frac{r}{2})$ , and so, for every  $r_0 \in (0, r)$  the absolute mean curvature function of  $\mathcal{F} \cap B_N(p, r_0)$  is

also unbounded. Choose  $r_0 \in (0, r)$  sufficiently small so that  $r_0$  is less than the injectivity radius function of  $N$  at  $p$  and so that the geodesic spheres in  $B_N(p, r_0)$  centered at  $p$  have positive (not necessarily constant) mean curvature with respect to the inward pointing normal vector. By Theorem 1.3,  $\mathcal{F} \cap B_N(p, r_0)$  satisfies the curvature estimate  $|\sigma_{\mathcal{F}}|(q) d_N(q, p) \leq C$  for some constant  $C > 0$  independent of  $q \in B_N(p, r_0) - \{p\}$ , where  $|\sigma_{\mathcal{F}}|$  is given by (2) (see the proof of item 4 of Corollary 5.9 for a similar argument). It follows from this curvature estimate that blow-up rescalings of  $\mathcal{F} \cap B_N(p, r_0)$  of the form  $\lambda_n[\mathcal{F} \cap B_N(p, r_0)] \subset \lambda_n[B_N(p, r_0) - \{p\}]$  for  $\lambda_n \rightarrow \infty$ , have subsequences that converge to a weak CMC foliation  $\mathcal{F}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , which gives item (2A) of the corollary, except for the condition that the limit CMC foliation  $\mathcal{F}_\infty$  is non-flat, whose proof we postpone for the moment.

**Assertion 7.2** *For  $r_0 > 0$  small enough, the ball  $B_N(p, r_0)$  contains no complete  $H$ -surfaces whose two-sided cover is stable, for any value of  $H \in \mathbb{R}$ .*

*Proof.* Fix  $r_0 > 0$  small, satisfying the properties previous to the assertion. By the main theorem in [23], there exists a uniform bound for the second fundamental form of any complete  $H$ -surface in  $B_N(p, r_1)$  whose two-sided cover is stable, which is independent of the value of  $H$ . In particular,  $|H|$  is also bounded from above by a universal constant  $H' > 0$ . Now choose  $r_1 \in (0, r_0)$  small enough so that the distance spheres  $S_N^2(p, \delta)$ ,  $\delta \in (0, r_1]$ , all have mean curvature strictly greater than  $H'$ , and suppose that  $B(p, r_1)$  contains a complete  $H$ -surface  $\Sigma$  whose two-sided cover is stable. Then, there exists a point  $q$  in the closure of  $\Sigma$  which is at maximal distance from  $p$ , and a sequence of small intrinsic open disks  $D_n \subset \Sigma$  that converge to an open  $H$ -disk  $D_\infty$  passing through  $q$ . As  $D_\infty$  lies in the closure of  $\Sigma$ , then  $D_\infty$  lies at one side of the distance sphere  $S_N^2(p, d_N(p, q))$  at  $q$ . The mean comparison principle applied to  $D_\infty$  and  $S_N^2(p, d_N(p, q))$  gives a contradiction as the mean curvature of  $S_N^2(p, d_N(p, q))$  is strictly less than  $H$ . This proves the assertion, after relabeling  $r_1$  by  $r_0$ .  $\square$

**Assertion 7.3** *Take  $r_0 > 0$  as in Assertion 7.2. Let  $H_0 = H_0(r_0)$  be the supremum of the mean curvatures of the leaves of  $\mathcal{F}$  that intersect  $S_N^2(p, r_0)$ . Let*

$$\mathcal{A}(r_0) = \{L \text{ leaf of } \mathcal{F} \cap B_N(p, r_0) : |H_L| > H_0\},$$

*where  $|H_L|$  is the absolute mean curvature of  $L$ . Then, the closure  $\bar{L}$  in  $B_N(p, r_0)$  of any  $L \in \mathcal{A}(r_0)$  is a compact immersed surface.*

*Proof.* Take  $L \in \mathcal{A}(r_0)$ . As  $L$  is a leaf of a weak CMC foliation, item 4 of Corollary 5.9 implies that the weak  $H_L$ -lamination  $\bar{L} - \{p\}$  of  $B_N(p, r_0) - \{p\}$  extends across  $p$  to a weak  $H_L$ -lamination of  $B_N(p, r_0)$ , all whose leaves are complete (note that the extrinsic distance from the lamination  $\bar{L}$  to  $S_N^2(p, r_0)$  is positive). If  $L$  were not proper, then the lamination  $\bar{L}$  would contain a complete leaf whose two-sided cover is stable, which contradicts Assertion 7.2. Hence,  $L$  is proper and so,  $\bar{L}$  is a connected, compact immersed  $H_L$ -surface in  $B_N(p, r_0)$ .  $\square$

**Assertion 7.4** *With the notation of Assertion 7.3 and after possibly choosing a smaller value of  $r_0$ ,  $\bar{L}$  is a compact embedded surface in  $B_N(p, r_0)$  for every  $L \in \mathcal{A}(r_0)$ .*

*Proof.* First observe that if a codimension-one CMC foliation  $\widehat{\mathcal{F}}$  of a Riemannian manifold  $N$  is transversely orientable<sup>6</sup>, then the leaves of  $\mathcal{F}$  are embedded; this follows from the fact that locally around a point  $q$  of self-intersection of a leaf  $L$  of  $\mathcal{F}$  with itself,  $L$  consists of two small embedded disks tangent at  $q$  with non-zero opposite mean curvature vectors at  $q$ . This contradicts the fact that the mean curvature vector of  $L$  equals  $H_L \cdot (N_{\mathcal{F}})|_L$  (up to sign), where  $N_{\mathcal{F}}$  is a continuous unitary vector field on  $N$  orthogonal to the leaves of  $\mathcal{F}$ .

We now prove the assertion. Since  $B_N(p, r_0) - \{p\}$  is simply connected, then  $\mathcal{F} \cap [B_N(p, r_0) - \{p\}]$  is transversely oriented. Thus, Assertion 7.3 and the observation in the last paragraph imply that if  $L \in \mathcal{A}(r_0)$ , then  $\bar{L}$  is a compact immersed surface and  $L$  is embedded. So it remains to show that for  $r_0$  sufficiently small, every leaf  $L$  in  $\mathcal{A}(r_0)$  with  $p \in \bar{L}$  satisfies that  $\bar{L}$  is embedded at  $p$ . Arguing by contradiction, suppose that for each  $n \in \mathbb{N}$ , there exists a leaf  $L_n$  in  $\mathcal{A}(\frac{r_0}{n})$  such that the compact immersed surface  $\bar{L}_n$  is embedded outside  $p$  and  $p$  is a point of self-intersection of  $\bar{L}_n$ . In particular,  $\bar{L}_n$  has non-zero mean curvature and the mean curvature vectors of  $\bar{L}_n$  point in opposite directions at the two points of the abstract surface  $\bar{L}_n$  occurring at the self-intersection point  $p$ ; this means that we can consider two local intrinsic disks  $D_n, \widehat{D}_n \in \bar{L}_n$ , that intersect only at  $p$ , and which have opposite mean curvatures with respect to a fixed orientation of their common tangent planes at  $p$ .

In particular, for some  $n_0$  large, the surface  $\bar{L}_1$  intersected with  $B_N(p, \frac{r_0}{n_0})$  consists of two disks, each of whose boundary curves lies in  $S_N^2(p, \frac{r_0}{n_0})$ ; let  $D$  be one of these two disks. Choose  $n_1 > n_0$  such that the absolute mean curvature of  $\bar{L}_{n_1} \subset B_N(p, \frac{r_0}{n_1}) \subset B_N(p, \frac{r_0}{n_0})$  is greater than the absolute mean curvature of  $D$ . Then,  $D_{n_1}, \widehat{D}_{n_1} \in \bar{L}_{n_1}$  are tangent to  $D$  at  $p$  and lie on the same side of  $D$ , since  $\bar{L}_{n_1}$  is connected and lies in one of the two closed complements of  $D$  in  $B_N(p, \frac{r_0}{n_0})$ . But then at the point  $p$ , the disk  $D$  lies on the mean convex side of one of the disks  $D_{n_1}, \widehat{D}_{n_1}$ , which contradicts the mean curvature comparison principle since the absolute mean curvature of  $D$  is less than that of  $\bar{L}_{n_1}$ . This contradiction finishes the proof of the assertion.  $\square$

**Assertion 7.5** *Given  $L \in \mathcal{A}(r_0)$ , let  $\Delta(\bar{L})$  be the compact subdomain of  $B_N(p, r_0)$  bounded by the compact embedded surface  $\bar{L}$ . Then,  $\Delta(\bar{L})$  contains the point  $p$ .*

*Proof.* Arguing by contradiction, suppose that  $p \notin \Delta(\bar{L})$ . Note that the mean convex side of the compact embedded surface  $\bar{L}$  is  $\Delta(\bar{L}) - \bar{L}$  (this follows from considering the innermost distance sphere  $S_N^2(p, \delta)$  such that  $\bar{L} \subset B_N(p, \delta)$  and comparing the mean curvature vectors of  $\bar{L}$  and of  $S_N^2(p, \delta)$ ). As  $\mathcal{F}$  does not have any singularities in  $\Delta(\bar{L})$ , then there exists a leaf  $L_1$  of  $\mathcal{F}$  which

<sup>6</sup>A codimension-one foliation  $\widehat{\mathcal{F}}$  of a manifold  $N$  is called *transversely orientable* if  $N$  admits a continuous, nowhere zero vector field whose integral curves are transverse to the leaves of  $\widehat{\mathcal{F}}$ .



maximizes the mean curvature function of  $\mathcal{F}$  restricted to  $\Delta(\bar{L})$ . If  $L_1 = L$ , then Proposition 5.4 in [17] implies that  $L$  is stable, which contradicts Assertion 7.2. If  $L_1 \neq L$ , then we can apply the same Proposition 5.4 in [17] on the mean convex side of  $L_1$ , which is strictly contained in  $\Delta(\bar{L})$  to contradict Assertion 7.2 again. This proves the assertion.  $\square$

**Assertion 7.6** *For  $r_0$  sufficiently small, the closure  $\bar{L}$  of every  $L \in \mathcal{A}(r_0)$  is topologically a sphere.*

*Proof.* Consider the distance function in  $N$  to  $p$ ,  $d_p: B_N(p, r_0) \rightarrow [0, r_0)$ . By Assertion 7.5, one of the following two exclusive possibilities holds when  $r_0$  is sufficiently small:

(A)  $p \in \text{Int}(\Delta(\bar{L}))$  for every  $L \in \mathcal{A}(r_0)$  (or equivalently,  $L = \bar{L}$  for every  $L \in \mathcal{A}(r_0)$  by Assertion 7.5).

(B)  $p \in \bar{L}$  for every  $L \in \mathcal{A}(r_0)$ .

Suppose first that we are in case (A) and pick  $L \in \mathcal{A}(r_0)$ . The restriction  $f = (d_p)|_L$  is strictly positive as  $L$  is compact and  $p \notin L$ . For simplicity, we will assume that  $f$  is a Morse function, although this is not strictly necessary (also, one could perturb slightly  $f$  to a Morse function so that the argument that follows remains valid). Let  $\hat{f}: \Delta(L) \rightarrow \mathbb{R}$  be the restriction of  $d_p$  to  $\Delta(L)$ . If  $L$  is not topologically a sphere, then  $\Delta(L)$  is not a closed topological ball and thus, there exists a critical point  $q$  of  $f$  in  $L$  such that  $\hat{f}^{-1}(0, f(q) - \varepsilon]$  is not homeomorphic to  $\hat{f}^{-1}(0, f(q) + \varepsilon]$  for all sufficiently small  $\varepsilon > 0$ . At such a critical point  $q$ ,  $L$  is tangent to the distance sphere  $S_N^2(p, f(q))$  at  $q$  and the mean curvature vector of  $S_N^2(p, f(q))$  points outward from the mean convex side of  $L$  at  $q$  (as  $f$  is assumed to be a Morse function, then  $f$  has index 1 at  $q$ ). By contradiction, suppose that the assertion fails in this case (A). Thus there exists a sequence  $r_n \searrow 0$ , leaves  $L_n \in \mathcal{A}(r_n)$ , none of which is a sphere, and critical points  $q_n \in L_n$  of  $f_n = \hat{f}|_{L_n}$  such that the mean curvature vector of  $L_n$  at  $q$  points outward  $B_N(p, d_p(q_n))$  at  $q_n$ . After rescaling by  $l_n = \frac{1}{d_p(p_n)}$  and extracting a subsequence, the weak CMC foliations  $l_n \mathcal{F}_n$  converge as  $n \rightarrow \infty$  to a weak CMC foliation  $\mathcal{F}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , whose leaves have closures being spheres or planes by Theorem 1.1. Furthermore,  $\mathcal{F}_\infty$  contains a leaf  $L_\infty$  that passes through the limit point  $q_\infty \in \mathbb{S}^2(1)$  as  $n \rightarrow \infty$  of the points  $l_n q_n$ . In particular, the closure  $\overline{L_\infty}$  of  $L_\infty$  is a plane or a sphere passing through  $q_\infty$ . As  $L_n$  is tangent to  $S_N^2(p_n, d_p(q_n))$  at  $q_n$ , then  $L_\infty$  is tangent to  $\mathbb{S}^2(|q_\infty|)$  at  $q_\infty$ . Therefore, we have three possibilities for  $L_\infty$ :

1.  $L_\infty$  is a plane passing through  $q_\infty$ , orthogonal to this position vector.
2.  $L_\infty$  is a sphere passing through  $q_\infty$  with mean curvature vector pointing in the direction of  $q_\infty$ .
3.  $\overline{L_\infty}$  is a sphere passing through  $q_\infty$  with mean curvature vector pointing in the direction of  $-q_\infty$ .

Also, observe that as the mean curvature vectors of  $L_n$ ,  $S_N^2(p, d_p(q_n))$  at their common point  $q_n$  point in opposite directions, then Case 3 above cannot occur. Cases 1, 2 cannot occur either, as then the distance function in  $\mathbb{R}^3$  to  $q_\infty$  has a non-degenerate (global) minimum at  $q_\infty$ , which contradicts that  $f_n$  has index 1 at  $q_n$  for all  $n$ . Hence the assertion holds for  $r_0$  sufficiently small in case (A).

In case (B) the argument is similar to the previous one, and we leave the details to the reader. This finishes the proof of the assertion.  $\square$

We next finish the proof of Theorem 7.1. Assertions 7.5 and 7.6 imply that item (2B) of the corollary holds. It only remains to show that given  $l_n > 0$  with  $l_n \rightarrow \infty$ , the weak CMC foliation  $\mathcal{F}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$  obtained as a limit of a subsequence of  $l_n[\mathcal{F} \cap B_N(p, r_0)]$  in the first paragraph of the proof of this corollary, is non-flat. Fix  $r_0 > 0$  small enough so that Assertion 7.6 holds.

**Assertion 7.7** *For  $n$  large enough, there exists  $L \in \mathcal{A}(r_0)$  such that  $\Delta(L) \subset \overline{B}_N(p, \frac{1}{l_n})$  and  $L \cap S_N^2(p, \frac{1}{l_n}) \neq \emptyset$ .*

*Proof.* Assume that  $n$  is chosen sufficiently large so that  $\frac{1}{l_n} < r_0$  and so that any leaf of  $\mathcal{A}(r_0)$  that is contained in  $\overline{B}_N(p, \frac{1}{l_n}) \subset \overline{B}_N(p, r_0)$  has mean curvature greater than  $2H_0$  (recall that  $H_0$  was defined in the statement of Assertion 7.3); this choice of  $n$  is possible since every leaf in  $\mathcal{A}(r_0)$  contained in  $\overline{B}_N(p, \frac{1}{l_n})$  is compact embedded sphere and for  $n$  large, the mean curvature such a leaf is at least  $\frac{1}{2l_n}$ .

Consider the non-empty open set

$$A_- = \bigcup_{L \in \mathcal{A}_-} \text{Int}[\Delta(L)], \quad \text{where } \mathcal{A}_- = \{L \in \mathcal{A}(r_0) \mid \Delta(L) \subset \overline{B}_N(p, \frac{1}{l_n})\}.$$

We claim that there exists  $L_1 \in \mathcal{A}(r_0)$  such that  $L_1 \subset \partial(A_-)$  and  $\Delta(L_1) \subset \overline{B}_N(p, \frac{1}{l_n})$ . To see this, take a point  $q \in \partial(A_-) - \{p\}$  and a sequence of points  $q_k \in A_-$  converging to  $q$  as  $k \rightarrow \infty$ . Then,  $q_k$  lies in the interior of  $\Delta(L_k)$  for some  $L_k \in \mathcal{A}(r_0)$  with  $\Delta(L_k) \subset \overline{B}_N(p, \frac{1}{l_n})$ . As the norm of the second fundamental form of leaves of  $\mathcal{F}$  is uniformly bounded locally around  $q$ , then all  $L_k$  can be locally expressed around  $q$  as graphs of uniform size over their tangent planes at  $q_k$ . By the Arzelà-Ascoli theorem, a subsequence of these graphs converges to a graph of constant mean curvature at least  $2H_0$ , and a monodromy argument shows that this limit graph is contained in a spherical leaf  $L_1 \in \mathcal{A}(r_0)$  with  $\Delta(L_1) \subset \overline{B}_N(p, \frac{1}{l_n})$ , so our claim is proved.

We will show that  $L_1$  intersects  $S_N^2(p, \frac{1}{l_n})$  (so the assertion will be proved by taking  $L = L_1$ ). Arguing by contradiction, suppose  $L_1 \cap S_N^2(p, \frac{1}{l_n}) = \emptyset$ . Now consider the family

$$\mathcal{A}_+ = \{L \in \mathcal{A}(r_0) \mid L - \overline{B}_N(p, \frac{1}{l_n}) \neq \emptyset \text{ and } \text{Int}(\Delta(L)) \cap \Delta(L_1) \neq \emptyset\},$$

which we claim is non-empty. The arguments that demonstrate that this set is non-empty are more or less the same as the ones given in the previous paragraph; one shows that for some

point  $q \in L_1 - \{p\}$  there is a sequence of points  $q_k \in \overline{B}_N(p, \frac{1}{l_n}) - \Delta(L_1)$  converging to  $q$  and contained in respective leaves  $L_k$  with mean curvatures converging to the mean curvature of  $L_1$ , and so the leaves  $L_k$  lie in  $\mathcal{A}_+$  for  $k$  large.

Since the family of balls  $\{\Delta(L) \mid L \in \mathcal{A}(r_0) \text{ and } \text{Int}(\Delta(L)) \cap \Delta(L_1) \neq \emptyset\}$  is totally ordered under inclusion, it follows that for every  $s \in \mathbb{N}$  and  $L^1, \dots, L^s \in \mathcal{A}_+$ , we have  $\bigcap_{i=1}^s [\Delta(L^i) \cap S_N^2(p, \frac{1}{l_n})] \neq \emptyset$ . Since  $S_N^2(p, \frac{1}{l_n})$  is compact, we conclude that there exists a point  $Q$  such that

$$Q \in \bigcap_{L \in \mathcal{A}_+} [\Delta(L) \cap S_N^2(p, \frac{1}{l_n})]. \quad (6)$$

As  $\mathcal{F}$  is a weak CMC foliation, there exists a leaf  $L_Q \in \mathcal{F}$  passing through  $Q$ , although  $L_Q$  might not be unique. Let  $\mathcal{F}_Q$  be the collection of leaves of  $\mathcal{F}$  passing through  $Q$ . Since  $S_N^2(p, \frac{1}{l_n}) \subset \text{Int}[\Delta(L')]$  and  $Q \in S_N^2(p, \frac{1}{l_n})$ , then every such leaf  $L_Q \in \mathcal{F}_Q$  lies in  $\mathcal{A}(r_0)$  (and thus, the closure of  $L_Q$  is a sphere). Also, every two leaves  $L_Q, L'_Q \in \mathcal{F}_Q$  intersect tangentially, with one at one side of the other; therefore either  $\Delta(L_Q) \subset \Delta(L'_Q)$  or vice versa. As there exists a uniform local bound around  $Q$  for the norms of the second fundamental forms of leaves in  $\mathcal{F}_Q$ , we conclude that the union of the leaves in  $\mathcal{F}_Q$  is a compact set of  $N$ . Thus, we can choose  $\tilde{Q} \in B_N(p, \frac{1}{l_n})$  with the following properties:

(P1)  $d_N(Q, \tilde{Q}) < \frac{\delta}{2}$ , where  $\delta = d(Q, L_1) = d(Q, A_-) > 0$ .

(P2)  $\tilde{Q} \in \text{Int}[\Delta(L_Q)]$ , for all  $L_Q \in \mathcal{F}_Q$ .

As before, there exists  $\tilde{L} \in \mathcal{A}(r_0)$  such that  $\tilde{Q} \in \tilde{L}$ . Note that  $Q \notin \tilde{L}$  (otherwise  $\tilde{L} \in \mathcal{F}_Q$ , which contradicts (P2) above). Therefore, (P2) implies that  $Q \notin \Delta(\tilde{L})$ . In turn, this implies by (6) that  $\tilde{L} \notin \mathcal{A}_+$ . As  $\tilde{L} \in \mathcal{A}(r_0)$ , then by definition of  $\mathcal{A}_+$  we have  $\tilde{L} \subset \overline{B}_N(p, \frac{1}{l_n})$  and so,  $\Delta(\tilde{L}) \subset \overline{B}_N(p, \frac{1}{l_n})$ . By definition of  $\mathcal{A}_-$ , this means that  $\tilde{L} \in \mathcal{A}_-$ , from where  $\text{Int}[\Delta(\tilde{L})] \subset A_-$ . This is impossible, as

$$\delta = d(Q, A_-) \leq d(Q, \tilde{L}) \leq d(Q, Q'),$$

which contradicts (P1). This contradiction finishes the proof of the assertion.  $\square$

**Assertion 7.8** *Item (2A) of Theorem 7.1 holds.*

*Proof.* Consider a sequence  $l_n > 0$  with  $l_n \rightarrow \infty$ . As we have explained, there exists  $r_0 > 0$  small such that after extracting a subsequence, the weak CMC foliations  $l_n[\mathcal{F} \cap B_N(p, r_0)]$  converge as  $n \rightarrow \infty$  to a weak CMC foliation  $\mathcal{F}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , and it only remains to show that  $\mathcal{F}_\infty$  is not flat. Consider the every  $n \in \mathbb{N}$  the distance sphere  $S_N^2(p, \frac{1}{l_n})$ . By Assertion 7.7, for  $n$  sufficiently large there exists  $L_n \in \mathcal{A}(r_0)$  such that  $L_n \subset \overline{B}_N(p, \frac{1}{l_n})$  and  $L_n \cap S_N^2(p, \frac{1}{l_n}) \neq \emptyset$ . Thus, the rescaled leaves  $l_n L_n$  of the weak CMC laminations  $l_n[\mathcal{F} \cap B_N(p, r_0)]$  stay inside the closed ball  $\overline{B}_{l_n N}(p, 1)$  with a point in  $S_{l_n N}^2(p, 1)$ , and so, the limit foliation  $\mathcal{F}_\infty$  has a leaf contained in the closed unit ball of  $\mathbb{R}^3$  with some point in the unit sphere; in particular,  $\mathcal{F}_\infty$  is non-flat.  $\square$

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